Journal of Cryptographic Engineering (2019) 9:115–136
https://doi.org/10.1007/s13389-018-0196-7

REGULAR PAPER

Efficient Fixed-base exponentiation and scalar multiplication based on a multiplicative splitting exponent recoding

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Received: 7 June 2017 / Accepted: 29 October 2018 / Published online: 12 November 2018

Abstract
Digital signature algorithm (DSA) (resp. ECDSA) involves modular exponentiation (resp. scalar multiplication) of a public and known base by a random one-time exponent. In order to speed up this operation, well-known methods take advantage of the memorization of base powers (resp. base multiples). Best approaches are the Fixed-base radix-R method and the Fixed-base Comb method. In this paper, we present a new approach for storage/online computation trade-off, by using a multiplicative splitting of the digits of the exponent radix-R representation. We adapt classical algorithms for modular exponentiation and scalar multiplication in order to take advantage of the proposed exponent recoding. An analysis of the complexity for practical size shows that our proposed approach involves a lower storage for a given level of online computation. This is confirmed by implementation results showing significant memory saving, up to 3 times for the largest NIST standardized key sizes, compared to the state-of-the-art approaches.

Keywords RNS · Multiplicative splitting · Digital signature · Fixed-base · Modular exponentiation · Scalar multiplication · Memory storage · Efficient software implementation

1 Introduction

In the digital signature standard (DSS), digital signature algorithm (DSA) is a popular authentication protocol. According to the NIST standard (see [11]), the public parameters are p, q and g. The parameter g is a generator of a multiplicative subgroup of $\mathbb{F}_p^*$ of size q. The integers p and q are two primes with sizes corresponding to the required security level: for the recommended security level 80–256 bits, q has to be a 160–512 bit integer. When a server needs to sign a batch of documents, the most costly operations are modular exponentiations $g^k \bmod p$ (one per signature), where g, p are fixed and k is a one-time random integer.

Another popular standard for electronic signature is ECDSA which uses the group of point on an elliptic curve $(E(\mathbb{F}_p), +)$ instead of $(\mathbb{F}_p^*, \times)$. The signature algorithm ECDSA is very similar to the DSA, and its main operation is a scalar multiplication $k \cdot P$ for $P \in E(\mathbb{F}_p)$. In order to cover both cases DSA and ECDSA we consider a multiplicative abelian group $(G, \times)$ in which we have to compute g^k for $g \in G$ and $k \in \mathbb{N}$.

In this article we consider the following practical case: a server has to compute a large number of signatures, which involves a large number of exponentiations g^k with the same $g \in G$ and several random k. We assume that the server has a large cache and RAM (random-access memory) so that we can therefore store a large amount of precomputed data to speed up these exponentiations. In the sequel, by 'offline computation' we mean the data computed only once and used in every signature generation; by 'online computation' we mean the operations required only in a single exponentiation g^k for a given k.

The main known methods of the state of the art which take advantage of large amount of precomputed data are the *Fixed-base Radix R* presented by Gordon [7] and the *Fixed-base Comb* presented by Lim and Lee in [13]. The *Fixed-base Radix R* method of [7] precomputes g^{aR^i} for $0 \leq a < R$ and then, using the radix-R expression of k, we obtain the exponentiation g^k with $\log_R(k)$ multiplications. The *Fixed-base*

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Comb method uses a Comb decomposition of k (instead of a radix-R representation) and requires less precomputed data at the cost of some extra squarings. In [16] the authors provide a variant of the Radix-R approach using the NAF_w recoding resulting in a reduced number of online multiplications than for the radix-R approach but with a penalty of some extra squarings.

Contributions We investigate some new strategies for a better trade-off between storage and online computation in Fixed-base exponentiation. To reach this goal, we propose to use the representation of the exponent in radix R as $k = \sum_{i=0}^{\ell-1} k_i R^i$ and then compute a multiplicative splitting of each digit k_i. Specifically, we use a radix $R = m_0 m_1$ with pairwise prime m_0, m_1. An RNS representation of a digit $k_i \in [0, R[$ in $\{m_0, m_1\}$ leads to a splitting into two parts: one part $k_i^{(0)}$ which value is at most m_0 and the other $k_i^{(1)}$ which value is at most m_1. We apply this process to all the digits of the radix R representation of the exponent. While processing the exponentiation, the digits $k_i^{(1)}$ are handled with a look-up table and the digits $k_i^{(0)}$ are handled with online computation. This approach was part of a preliminary version of this paper published in the proceedings of the WAIFI 2016 conference [19].

We present a novel approach for the multiplicative splitting of the digits of the exponent: if we choose the radix R as a prime integer, then processing a partial execution of the extended euclidean algorithm, one can re-express a digit k_i as product $k_i = k_i^{(0)}(k_i^{(1)})^{-1} \mod R$ where $|k_i^{(1)}| < c$ and $|k_i^{(0)}| < R/c$ for a fixed c. Again, this splitting can be applied to all digits of the radix R representation of the exponent. The exponentiation algorithms can then be computed with memorizations related to the $(k_i^{(1)})^{-1}$ part of the digit splitting and online computation to handle the part $k_i^{(0)}$ of the digit splitting. The main advantage of this version with a prime R is that the resulting exponentiation algorithm is constant time, which means that it is robust against timing attacks.

We study the corresponding complexities and storage amounts, and compare the results with the best approaches of the literature for Fixed-base modular exponentiation (resp. scalar multiplication) for NIST recommended fields (resp. curves). The metric chosen for a comparison between the proposed algorithms is the following: for a given level of online computation the best approach is the one which has the lowest amount of precomputed data. Using this metric we show that the proposed approach is the more efficient for a large range of practical case. We also implement these approaches in software and we perform tests in order to validate the complexity analysis. Our approaches provide also some flexibility in terms of required storage amount: one can choose the storage amount according to the device resources available and compatible to the global computation load of the system.

Organization of the paper In Sect. 2, we review the best approaches of the literature for Fixed-base exponentiation and we give their complexities and storage requirements. In Sect. 3, we present a multiplicative splitting recoding of the exponent in radix $R = m_0 m_1$ and a Fixed-base exponentiation using this recoding. In Sect. 4, we present a multiplicative splitting recoding for R prime and the corresponding exponentiation algorithm. In Sect. 5, we compare the complexity results and software implementations of the proposed approach to the best approaches of the literature for modular exponentiation and scalar multiplication. Finally, in Sect. 6, we give some concluding remarks and perspectives.

2 State of the art of Fixed-base exponentiation

We consider digital signature algorithms based on discrete logarithm in a finite group. The main ones are DSA where the considered group is a subgroup of prime order q in the multiplicative group $\mathbb{F}_p^*$ and ECDSA where the group is the set of point on an elliptic curve $E(\mathbb{F}_p)$ [12,15]. For the sake of simplicity, in the sequel, we use a generic abelian multiplicative group $(G, \times)$ of order q. The algorithms presented later in this paper extend directly to abelian groups with additive group law like $E(\mathbb{F}_p)$. Generating a digital signature consists in computing (s_1, s_2) from a message $m \in \{0,1\}^*$, a secret integer x and a random integer k as follows

$$s_1 \leftarrow H_1(g^k),$$
$$s_2 \leftarrow (H_2(m) + s_1 x)k^{-1} \mod q.$$

Here, H_1 is a function $G \to \mathbb{Z}/q\mathbb{Z}$ and H_2 is a cryptographic hash function $\{0,1\}^* \to \mathbb{Z}/q\mathbb{Z}$. One can see that the most costly operation in a signature generation is the exponentiation g^k of a fixed $g \in G$ and where k is a one-time random exponent of size $\cong q$. This exponentiation can be done with the classical Square-and-multiply algorithm.

Square-and-multiply exponentiation The left-to-right version of the square-and-multiply exponentiation scans the bits k_i of k from left to right and performs a squaring followed by a multiplication when $k_i = 1$. In terms of complexity, given the bit length t of k, the number of squarings is $t-1$ and the number of multiplications to be computed is $t/2$ on average for a randomly chosen exponent. There is no storage in this case.

Side-channel analysis The above method is threatened by side-channel analysis. These attacks extract part of the exponent by monitoring and analyzing the computation time, the power consumption or the electromagnetic emanations. In this paper, we focus on servers which generate large amounts of signature very quickly and are physically not accessible to an attacker. The main threat in this case is the timing attack. This attack attempts to find the sequence of operations (multiplication and squaring) of an exponentiation by

Algorithm 1 Left-to-Right Square-and-multiply Exponentiation

Require: Let an integer $k = (k_{t-1}, \ldots, k_0)_2$, and g an element of G.
Ensure: $X = g^k$
1: $X \leftarrow 1$
2: **for** i **from** $t-1$ **downto** 0 **do**
3: $\quad X \leftarrow X^2$
4: $\quad$ **if** $k_i = 1$ **then**
5: $\quad\quad X \leftarrow X \cdot g$
6: **return** (X)

a statistical analysis of several timings of an exponentiation. If the assumed sequence of operations is correct, the attacker can deduce the key bits of the exponent since each multiplication corresponds to a bit equal to 1, otherwise the bit is 0. A general solution to thwart this attack is to render the sequence of operations not correlated to key bits, which means that we need to remove any **if** test on the key bits or digits in the exponentiation algorithm.

Fixed-base exponentiation When the base g is fixed, one can precompute in advance some data in order to reduce the number of operations in the online computation of the exponentiation. This is the case when a server has to intensively compute a number of signatures with the same g. For example, the method presented by Gordon in [7] is a modified square-and-multiply algorithm: one first stores the t successive squarings of g (that is the sequence of g^{2^i}), then for a given computation of g^k, one has to multiply the g^{2^i} corresponding to $k_i = 1$. In terms of complexity, given the bit length t of the exponent, one has now no squarings and the number of multiplications is $t/2$, in average. As counterpart, one has to store t elements of G. We can even further reduce the amount of online computation by increasing the precomputed data. This is the strategy followed by the main approaches of the literature.

Radix-R method Gordon in [7] mentions the generalization of his first idea to radix $R = 2^w$ representation of the exponent $k = \sum_{i=0}^{\ell-1} k_i R^i$. This consists in the memorization of the values $g^{a \cdot R^j}$, with $a \in [0, \ldots, R-1]$ and $0 \leq j < \ell$ where ℓ is the length of the exponent in radix R representation. If we denote $w = \lceil \log_2(R) \rceil$, then we have $\ell = \lceil t/w \rceil$. In this case, the online computation consists of $\ell - 1$ multiplications, for a storage amount of $\ell \cdot R$ values in G. In the sequel, we will call this approach the *Fixed-base Radix-R* exponentiation method (see Algorithm 2). This algorithm is constant time as soon as the multiplications by 1 (i.e., when $k_i = 0$) are performed as any other multiplication or, alternatively, by using the radix R recoding of [10] which avoids $k_i = 0$.

Comb method Another classical method is the so-called *Fixed-base Comb* method which was initially proposed by Lim and Lee in [13]. This method attempts to trade some

Algorithm 2 Fixed-Base Radix-R Exponentiation

Require: $k = (k_{\ell-1}, \ldots, k_0)_R$, g a generator of G.
Ensure: $X = g^k$
1: *Offline precomputation.* Store $T[a][j] \leftarrow g^{a \cdot R^j}$, with $a \in [0, ..., R-1]$ and $0 \leq j < \ell$.
2: $X \leftarrow 1$
3: **for** i **from** $\ell - 1$ **downto** 0 **do**
4: $\quad X \leftarrow X \cdot T[k_i][i]$
5: **return** (X)

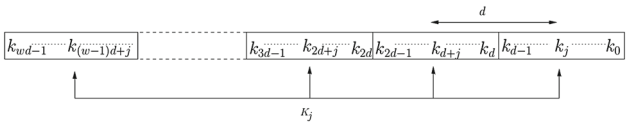
Algorithm 3 Fixed-Base Comb Exponentiation [13]

Require: $k = (k_{t-1}, \ldots, k_1, k_0)_2$, a generator g of G, a window width 2^w and $d = \lceil t/w \rceil$.
Ensure: $X = g^k \bmod p$
1: *Offline precomputation.* For all $(a_{w-1}, \ldots, a_0) \in \{0,1\}^w$ we set $a = a_{w-1}2^{(w-1)d} + \cdots + a_1 2^d + a_0$ and $T[(a_{w-1}, \ldots, a_0)_2] = g^a$.
2: Split $k = \sum_{j=0}^{d-1} K_j 2^j$ as in (1)
3: $X \leftarrow 1$
4: **for** j **from** $d-1$ **downto** 0 **do**
5: $\quad X \leftarrow X^2$
6: $\quad X \leftarrow X \cdot T[K_j]$
7: **return** (X)

of the storage of Algorithm 2 with a few online computed squarings. It is based on the following decomposition of the exponent k

$$k = \sum_{j=0}^{d-1} \underbrace{\left(\sum_{i=0}^{w-1} k_{id+j} 2^{id} \right)}_{K_j} 2^j \text{ where } d = \lceil t/w \rceil. \tag{1}$$

Each integer K_j can be seen as a comb as described in the following diagram.



The integer w is the number of comb-teeth in each K_j and $d = \lceil t/w \rceil$ is the distance in bits between two consecutive teeth. When all the possible values g^{K_j} are precomputed and stored in table indexed by

$$I_{K_j} = [k_{(w-1)d+j} k_{(w-2)d+j} \ldots, k_j]_2,$$

one can compute g^k with a 2^w size look-up table, $\lceil t/w \rceil - 1$ multiplications and $\lceil t/w \rceil - 1$ squarings using (1). This method is shown in Algorithm 3. As in the case of Radix-R method, this approach can be implemented in constant time if the multiplications by 1 (which occurs $K_j = 0$) are computed as an arbitrary multiplication or by using the recoding of [9] which renders all comb coefficients $\neq 0$.

Fixed-base exponentiation with NAF_w In [16], the authors proposed an alternative approach when inverting an element in the group G is almost free of computation and

multi-squarings can be computed efficiently. Their main application is the group of points on a elliptic curves where computing the inverse of a point is really cheap. They use a NAF_w representation of k in order to reduce the number of multiplications (this generalizes the approach of [21] which uses a NAF representation of k). Specifically, they start by computing the NAF_w representation of the exponent k

$$k = k'_{t-1}2^{t-1} + k'_{t-2}2^{t-2} + \cdots + k'_0$$

where $k'_i \in \{\pm 1, \pm 3, \ldots, \pm 2^{w-1} - 1\}$ and there are at least w zero between two non zero coefficients. For more details on NAF_w the reader may refer to [8]. Then they rewrite this $NAF_w(k)$ into $\ell = \lceil t/w \rceil$ consecutive windows of w coefficients:

$$k = \sum_{i=0}^{\ell} \left(\underbrace{\sum_{j=0}^{w-1} k'_{iw+j} 2^j}_{K_i} \right) 2^{iw}. \tag{2}$$

In [16] the authors noticed that, in each K_i, there is at most one nonzero coefficient k'_{iw+j}, which means that $K_i = s \times a \times 2^j$ for some $s \in \{-1, 1\}$, $a \in \{1, 3, \ldots, 2^{w-1} - 1\}$ and $0 \le j < w$. They then reorder the terms in expression (2) by splitting the parameter i into two parts $i = i_1 e + i_0$ for some fixed integer e:

$$\begin{aligned} k &= \textstyle\sum_{i_0=0}^{e-1} \sum_{i_1=0}^{d-1} K_{i_1 e+i_0} 2^{i_1 ew + i_0 w} \text{ where } d = \lceil \ell/e \rceil \\ &= \textstyle\sum_{i_0=0}^{e-1} \left(\sum_{i_1=0}^{d-1} K_{i_1 e+i_0} 2^{i_1 ew} \right) 2^{i_0 w}. \end{aligned} \tag{3}$$

For all possible values for $K_{i_1 e+i_0} 2^{i_1 ew}$ with $K_{i_1 e+i_0} = sa2^j$ the term $g^{a2^{j+i_1 ew}}$ is stored in a Table $T[a][i_1][j]$. Then Algorithm 4 computes g^k based on (3) as a sequence of multiplications/divisions (in Step 9 depending on $s = 1$ or $s - 1$) and w consecutive squarings (in Step 5).

Algorithm 4 Fixed-Base Exponentiation with NAF_w [16]

Require: A scalar $k = (k'_{t-1}, \ldots, k'_1, k'_0)_{NAF_w}$ and g in an abelian group G, and positive integers c, w.
Ensure: $X = g^k$
1: $\ell = \lceil \frac{t}{w} \rceil$ and $d = \lceil \frac{\ell}{e} \rceil$
2: *Offline precomputation.* $T[a][i_1][j] = g^{a2^{j+ewi_1}}$ for all $a \in \{1, 3, \ldots, 2^{w-1} - 1\}$, $i_1 \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, w-1\}$.
3: $X \leftarrow 1$
4: **for** i_0 **from** $e-1$ **downto** 0 **do**
5: $\quad X \leftarrow X^{2^w}$
6: $\quad$ **for** i_1 **from** $d-1$ **downto** 0 **do**
7: $\quad\quad (s, a, j)$ s.t. $(k'_{jb+t,w-1} \ldots k'_{jb+t,0})_{NAF_w} = s \cdot a \cdot 2^j$
8: $\quad\quad$ **if** $a \neq 0$ **then**
9: $\quad\quad\quad X \leftarrow X \times (T[a][i_1][j])^s$
10: **return** (X)

In Algorithm 4 the number of precomputed elements is equal to $dw2^{w-2} \cong \lceil \frac{t}{e} \rceil 2^{w-2}$. The online computation consists of $w(e-1)$ squarings and $ed(1 - (\frac{w}{w+1})^w) \cong \lceil \frac{t}{w} \rceil (1 - (\frac{w}{w+1})^w)$ multiplications/divisions (cf. [16] for details).

3 Fixed-base exponentiation with multiplicative splitting with $R = m_0 m_1$

We now present our approach of a Fixed-base exponentiation with multiplicative splitting with $R = m_0 m_1$. In this section, we review the method presented in a preliminary work at WAIFI 2016 [19]. The goal is to use a multiplicative splitting of the digits of k in order to provide a better trade-off between storage and online computation in the exponentiation.

3.1 Digit multiplicative splitting for radix $R = m_0 m_1$

A natural way to get a splitting of the digits is to use the RNS representation in radix $R = m_0 \cdot m_1$ which splits any digit into two parts. When all the digits of an exponent are split, we can process the exponentiation as follows: the first part of the digits will be used to select the precomputed values and the second part will be processed by online computation.

We first remind the RNS representation in a base $\mathcal{B} = \{m_0, m_1\}$. Let $R = m_0 \cdot m_1$ and $x \in \mathbb{Z}$ such that $0 \le x < R$. Let us also assume m_0 is prime, since this allows us to invert all nonzero integers $< m_0$ modulo m_0, and we choose $m_1 < m_0$. In the sequel, we denote $|x|_m = x \bmod m$.

One represents x with the residues

$$\begin{cases} x^{(0)} = |x|_{m_0}, \\ x^{(1)} = |x|_{m_1}, \end{cases}$$

and x can be retrieved using the Chinese Remainder Theorem as follows:

$$x = \left| x^{(0)} \cdot m_1 \cdot |m_1^{-1}|_{m_0} + x^{(1)} \cdot m_0 \cdot |m_0^{-1}|_{m_1} \right|_R. \tag{4}$$

We now present our recoding approach. We consider an exponent k expressed in radix $R = m_0 \cdot m_1$

$$k = \sum_{i=0}^{\ell-1} k_i R^i \text{ with } \ell = \lceil t / \log_2(R) \rceil.$$

We represent every radix-R digit in RNS with the RNS base $\mathcal{B} = \{m_0, m_1\}$: if k_i is the i-th digit of k in radix-R, we denote by $(k_i^{(0)}, k_i^{(1)})$ its RNS representation in base $\mathcal{B}$

$$\begin{cases} k_i^{(0)} = |k_i|_{m_0}, \\ k_i^{(1)} = |k_i|_{m_1}. \end{cases}$$

Let us denote

$$m'_0 = m_1 \cdot |m_1^{-1}|_{m_0},$$
$$m'_1 = m_0 \cdot |m_0^{-1}|_{m_1}.$$

We recode the digits of k in $\mathcal{B} = \{m_0, m_1\}$ as follows

- *If* $k_i^{(1)} \neq 0$: we denote

$$\begin{cases} k_i'^{(0)} = |k_i^{(0)} \cdot (k_i^{(1)})^{-1}|_{m_0}, \\ k_i'^{(1)} = k_i^{(1)}. \end{cases}$$

One keeps

$$k'_i = k_i'^{(1)} | k_i'^{(0)} \cdot m'_0 + m'_1 |_R \tag{5}$$

as a representation of k_i in a multiplicative splitting form and we have $k_i = |k'_i|_R$ with (4). When modifying the digits of k as above, one needs to take into account the correcting term due to the reduction modulo R:

$$k_i = k_i'^{(1)} | k_i'^{(0)} \cdot m'_0 + m'_1 |_R - \lfloor k_i'^{(1)} \cdot | k_i'^{(0)} \cdot m'_0 + m'_1 |_R / R \rfloor \cdot R.$$

Let us denote $C = \lfloor k_i'^{(1)} \cdot (k_i'^{(0)} \cdot m'_0 + m'_1)/R \rfloor$ which satisfies $0 \leq C < m_1$. We consider C as a carry that one can subtract to k_{i+1}. This leads to the following computation

if $k_{i+1} \geq C$ **then**
  $k_{i+1} \leftarrow k_{i+1} - C$
  $C \leftarrow 0$
else
  $k_{i+1} \leftarrow k_{i+1} + R - C$,
  $C \leftarrow 1$

and one gets $k_{i+1} \geq 0$.

- *If* $k_i^{(1)} = 0$: we define k'_i as follows

$$k'_i = \underbrace{\left| |k_i^{(0)} + 1|_{m_0} \cdot m'_0 + m'_1 \right|_R}_{(*)} - \underbrace{|m'_0 + m'_1|_R}_{=1}. \tag{6}$$

and k'_i satisfies $|k'_i|_R = k_i$ This expression is meant to have the part $(*)$ as in (5): the goal is to use the same precomputed data in the exponentiation algorithm. The term $-|m'_0 + m'_1|_R = -1$ is meant to get back to k_i while reducing k'_i modulo R. We then set the following coefficients:

$$\begin{cases} k_i'^{(0)} = |k_i^{(0)} + 1|_{m_0}, \\ k_i'^{(1)} = 0. \end{cases}$$

Setting $k_i'^{(1)} = 0$ tells us that this is a special case and we get k_i from $k_i'^{(0)}$ as

$$k_i = \left| |k_i'^{(0)} \cdot m'_0 + m'_1)|_R - 1 \right|_R.$$

We deal with the carry as it was done when $k_i^{(1)} \neq 0$, this is detailed in the algorithm.

One notices it might be necessary to handle the last carry C generated by the recoding of $k_{\ell-1}$ with a final correction. This gives a final coefficient $k'_\ell = -C$ which satisfies $|k'_\ell| < m_1$. Finally, this leads to the recoding algorithm shown in Algorithm 5.

Algorithm 5 Multiplicative Splitting Recoding with $R = m_0 m_1$

Require: An RNS base $\{m_0, m_1\}$, a radix $R = m_0 \cdot m_1$ and an exponent $k = \sum_{i=0}^{\ell-1} k_i R^i$.
Ensure: $\{(k_i'^{(0)}, k_i'^{(1)}), 0 \leq i < \ell, (C)\}$ the multiplicative splitting recoding of k in radix $R = m_0 m_1$.

$C \leftarrow 0$
for i **from** 0 **to** $\ell - 1$ **do**
  $k_i \leftarrow k_i - C, C \leftarrow 0$
  if $k_i < 0$ **then**
    $k_i \leftarrow k_i + R, C \leftarrow 1$
  $k_i^{(0)} \leftarrow |k_i|_{m_0}, k_i^{(1)} \leftarrow |k_i|_{m_1}$.
  if $k_i^{(1)} = 0$ **then**
    $(k_i'^{(0)}, k_i'^{(1)}) \leftarrow (|k_i^{(0)} + 1|_{m_0}, 0)$
    $C \leftarrow C + \left\lfloor \left(|k_i'^{(0)} \cdot m'_0 + m'_1|_R - 1 \right) / R \right\rfloor$
  else
    $k_i'^{(0)} \leftarrow |k_i^{(0)} \cdot (k_i^{(1)})^{-1}|_{m_0}$
    $k_i'^{(1)} \leftarrow k_i^{(1)}$
    $C \leftarrow C + \lfloor k_i'^{(1)} \cdot |k_i'^{(0)} \cdot m'_0 + m'_1|_R / R \rfloor$
return $\{(k_i'^{(0)}, k_i'^{(1)}), 0 \leq i < \ell, k'_\ell = -C\}$

At the end the recoded exponent $k = \sum_{i=0}^{\ell} k'_i R^i$ has most of its digits k'_i expressed as a product $k_i'^{(1)} \times |k_i'^{(0)} \cdot m'_0 + m'_1|_R$ and $k_i'^{(1)}$ is of size m_1 while $|k_i'^{(0)} \cdot m'_0 + m'_1|_R$ is indexed with $k_i'^{(0)}$ which is of size m_0.

Example 1 We present here an example of the $m_0 m_1$ recoding with an exponent size t of 20 bits ($0 < k < 2^{20}$), and $\mathcal{B} = \{11, 8\}$ (i.e., $m_0 = 11, m_1 = 8$). Thus, in this case, one has the radix $R = m_0 \cdot m_1 = 88$, $\ell = \lceil 20/\log_2(88) \rceil = 4$, and also

$$m'_0 = 8 \cdot |8^{-1}|_{11} = 56,$$
$$m'_1 = 11 \cdot |11^{-1}|_8 = 33.$$

Let us take $k = 936192_{10}$, the random exponent. By rewriting k in radix-R, one has

$$k = 48 + 78 \cdot 88 + 32 \cdot 88^2 + 1 \cdot 88^3.$$

We now use Algorithm 5, which consists of a `for` loop (Steps 2 to 13).

- In the first iteration ($i = 0$), one has $k_0 = 48$.
  - One has $C \leftarrow 0$ and one skips the `if`-test steps 4 to 5 since $k_0 \geq 0$.
  - Step 6, one computes the RNS representation in base $\mathcal{B}$ of $k_0 = 48$:

    $$k_0^{(0)} = |k_0|_{11} = 4, k_0^{(1)} = |k_0|_8 = 0.$$

  - Steps 7 to 9, since $k_0^{(1)} = 0$, one sets

    $$(k_0'^{(0)}, k_0'^{(1)}) \leftarrow (|k_0^{(0)} + 1|_{11}, 0) = (5, 0).$$

    and the carry

    $$C \leftarrow C + \left\lfloor \left(|k_0'^{(0)} \cdot 56 + 33|_{88} - 1 \right) / 88 \right\rfloor = 0$$

- In the second iteration ($i = 1$), one has $k_1 = 78$.
  - One has $C \leftarrow 0$ and one skips the `if`-test of Steps 4 to 5 since $k_1 \geq 0$.
  - Step 6, one computes the RNS representation in base $\mathcal{B}$ of $k_1 = 78$:

    $$k_1^{(0)} = |k_1|_{11} = 1, k_1^{(1)} = |k_1|_8 = 6.$$

  - Steps 10 to 13, since $k_1^{(1)} \neq 0$, one has

    $$\begin{array}{ll} |(k_1^{(1)})^{-1}|_{11} & \leftarrow 2 \\ k_1'^{(0)} = |k_1^{(0)} \cdot (k_1^{(1)})^{-1}|_{11} & \leftarrow 2 \\ k_1'^{(1)} = k_1^{(1)} & \leftarrow 6 \\ C \leftarrow \lfloor (k_1'^{(1)} \cdot |k_1'^{(0)} \cdot 56 + 33|_{88})/88 \rfloor & \leftarrow 3 \end{array}$$

- In the third iteration ($i = 2$), one has now $k_2 \leftarrow k_2 - C = 29$.
  - The RNS representation in base $\mathcal{B}$ of k_2 is $k_2^{(0)} = 7, k_2^{(1)} = 5$.
  - The Steps 10-13 give $C \leftarrow 2$, and

    $$(k_2'^{(0)}, k_2'^{(1)}) \leftarrow (8, 5).$$

Without providing all the remaining details, one finally obtains the values returned by the algorithm:

$((5, 0), (2, 6), (8, 5), (3, 7))$, and $k_4' = -C = -2$.

3.2 Exponentiation with a multiplicative splitting recoding in radix $R = m_0 m_1$

We first rewrite the exponentiation using the recoding of $k = \sum_{i=0}^{\ell} k_i' R^i$ of the previous subsection as follows:

$$\begin{aligned} g^k \bmod p &= g^{\sum_{i=0}^{\ell} k_i' \cdot R^i} \\ &= g^{k_\ell' \cdot R^\ell} \cdot \prod_{i=0}^{\ell-1} g^{k_i' \cdot R^i} \end{aligned} \tag{7}$$

where each term $g^{k_i' \cdot R^i}$ satisfies one of the following three cases:

- When $k_i'^{(1)} \neq 0$ and $i < \ell$:

  $$g^{k_i' \cdot R^i} = g^{k_i'^{(1)} \cdot R^i \cdot |k_i'^{(0)} \cdot m_0' + m_1'|_R}$$

- When $k_i'^{(1)} = 0$ and $i < \ell$:

  $$g^{k_i' \cdot R^i} = g^{R^i \cdot |k_i'^{(0)} \cdot m_0' + m_1'|_R} \cdot g^{-R^i}.$$

- when $i = \ell$ we have $k_\ell' \leq 0$ which implies that $g^{k_\ell' R^\ell} = (g^{-R^\ell})^{|k_\ell'|}$.

In order to compute the Fixed-base exponentiation g^k, one stores the following values:

$$T[i][j] = g^{R^i \cdot |j \cdot m_0' + m_1'|_R}, \text{ with } \begin{cases} 0 \leq i \leq \ell - 1, \\ 0 \leq j < m_0. \end{cases}$$

and one also stores the following inverses:

$$T[i][-1] = g^{-R^i} \text{ with } 0 \leq i \leq \ell.$$

We use Y_j to denote the product of $g^{R^i \cdot |k_i'^{(0)} \cdot m_0' + m_1'|_R}$ for each i such that $k_i'^{(1)} = j$. In other words for $j \neq 0$

$$Y_j = \begin{cases} \left(\prod_{\text{for } k_i'^{(1)} = j, i < \ell} T[i][k_i'^{(0)}] \right) \cdot T[\ell][-1] \text{ if } |k_\ell'| = j, \\ \left(\prod_{\text{for } k_i'^{(1)} = j, i < \ell} T[i][k_i'^{(0)}] \right), \end{cases}$$

and

$$Y_0 = \prod_{\text{for all } k_i'^{(1)} = 0, i < \ell} T[i][k_i'^{(0)}] \times T[i][-1].$$

We can then rewrite the expression of g^k in (7) in terms of Y_j for $j = 0, \ldots, m_1 - 1$ as follows:

$$g^k = Y_0 \times \prod_{j=1}^{m_1 - 1} Y_j^j.$$

Table 1 Example of an execution trace for an exponentiation based on multiplicative splitting recoding with $R = m_0 m_1$

Iter. (loop 3:)	Exp. coef.	Step	Value
$i = 0$	$k_0'^{(0)} = 5$	6:	$Y_0 \leftarrow T[0][k_0'^{(0)}] \times T[0][-1] = g^{49} \times g^{-1} = g^{48}$
	$k_0'^{(1)} = 0$		
$i = 1$	$k_1'^{(0)} = 2$	11:	$Y_6 \leftarrow T[1][k_1'^{(0)}] = g^{88 \cdot 57} = g^{5016}$
	$k_1'^{(1)} = 6$		
$i = 2$	$k_2'^{(0)} = 8$	11:	$Y_5 \leftarrow T[2][k_2'^{(0)}] = g^{88^2 \cdot 41} = g^{317504}$
	$k_2'^{(1)} = 5$		
$i = 3$	$k_3'^{(0)} = 3$	11:	$Y_7 \leftarrow T[3][k_3'^{(0)}] = g^{88^3 \cdot 25} = g^{17036800}$
	$k_3'^{(1)} = 7$		
-	-	15:	$T_2 \leftarrow T[4][-1] = g^{88^4 \cdot (-1)} = g^{-59969536}$
-	-	17: to 22:	$g^k = Y_0 \times \prod_{j=1}^{m_1-1} Y_j^j = g^{48} g^{2 \cdot (-59969536)} \times g^{5 \cdot 317504} \times g^{6 \cdot 5016} g^{7 \cdot 17036800} = g^{936192}$

Each individual exponentiation Y_j^j is performed with a square-and-multiply approach, which is more efficient than performing $j - 1$ multiplications, even for small m_1. This approach is depicted in Algorithm 6.

Algorithm 6 Fixed-base exponentiation with multiplicative splitting with radix $R = m_0 m_1$

Require: An RNS base $\{m_0, m_1\}$, a radix $R = m_0 m_1$, the exponent $k = \sum_{i=0}^{\ell-1} k_i R^i$ and $\{(k_i'^{(0)}, k_i'^{(1)}), 0 \le i < \ell, (k'_\ell)\}$ the $m_0 m_1$ recoding of k and $g \in G$.
Ensure: $A = g^k$
1: *Offline precomputation.* Store $T[i][j] \leftarrow g^{R^i \cdot |j \cdot m'_0 + m'_1|_R}$ with $0 \le i < \ell, 0 \le j < m_0, T[i][-1] \leftarrow g^{-R^i}, 0 \le i \le \ell$
2: $X \leftarrow 1, Y_j \leftarrow 1$ for $0 \le j < m_1$
3: **for** i **from** 0 **to** $\ell - 1$ **do**
4: **if** $k_i'^{(1)} = 0$ **then**
5: **if** $Y_0 = 1$ **then**
6: $Y_0 \leftarrow T[i][k_i'^{(0)}] \times T[i][-1]$
7: **else**
8: $Y_0 \leftarrow Y_0 \times T[i][k_i'^{(0)}] \times T[i][-1]$
9: **else**
10: **if** $Y_{k_i'^{(1)}} = 1$ **then**
11: $Y_{k_i'^{(1)}} \leftarrow T[i][k_i'^{(0)}]$
12: **else**
13: $Y_{k_i'^{(1)}} \leftarrow Y_{k_i'^{(1)}} \times T[i][k_i'^{(0)}]$
14: **if** $k'_\ell \neq 0$ **then**
15: $Y_{|k'_\ell|} \leftarrow Y_{|k'_\ell|} \times T[\ell][-1]$
16: $W \leftarrow$ size of m_1 in bits
17: **for** i **from** $W - 1$ **downto** 0 **do**
18: $X \leftarrow X^2$
19: **for** j **from** $m_1 - 1$ **downto** 1 **do**
20: **if** bit i of j is non zero **then**
21: $X \leftarrow X \times Y_j$
22: **return** $(X \times Y_0)$

One important drawback of the above algorithm is that it is not constant time, due to the **if** branching attached to the condition $k_i'^{(1)} = 0$.

Table 2 Hamming weights account for $0 \le j < m_1$

m_1	2	3	4	5	6	7	8	9
$\mathcal{H}$	1	2	4	5	7	9	12	13

Example 2 We present the computation of $g^k \bmod p$ using Algorithm 6, we take $\mathcal{B} = \{11, 8\}$ (i.e., $m_0 = 11, m_1 = 8$). In terms of storage, one computes the values

$$T[i][j] = g^{R^i \cdot |j \cdot m'_0 + m'_1|_R} \mod p \text{ with } 0 \le i \le \ell - 1.$$

One has the values $\{33, 1, 57, 25, 81, 49, 17, 73, 41, 9, 65\}$ for $|j \cdot m'_0 + m'_1|_R$ when $0 \le j < 11$. This leads to

$$T[i][0..10] = \{g^{88^i \cdot 33}, g^{88^i}, g^{88^i \cdot 57}, g^{88^i \cdot 25}, g^{88^i \cdot 81}, g^{88^i \cdot 49}, g^{88^i \cdot 17}, g^{88^i \cdot 73}, g^{88^i \cdot 41}, g^{88^i \cdot 9}, g^{88^i \cdot 65}\}.$$

The trace of Algorithm 6 for the computation of g^k and $k = 936192$ using the recoding obtained in Example 1 is provided in Table 1.

3.3 Complexity

For the amount of precomputed data, one can notice that it is equal to $(m_0 + 1) \times \ell + 1$ elements.

The complexity of online computation in Algorithm 6 is evaluated step by step in Table 3 for the average case. The number of multiplications (M) is evaluated as follows:

- The costs of Steps 6 to 15 follow directly from Algorithm 6 and are detailed in Table 3.
- The first squaring in Step 18 skipped since $X = 1$, leading to a cost of $W - 1$ squarings.
- The multiplications in Steps 21 and 22 are performed only in case of $Y_j \neq 1$. This means that in the worst case

we save the first multiplication which is an affectation : this is the case considered in Table 3.

For the sake of simplicity, we denote by $\mathcal{H}$ the sum of the j Hamming weights for each j from $m_1 - 1$ downto 1 (`for` loop in Step 19). The value of $\mathcal{H}$ is shown in Table 2 for different practical values of m_1.

4 Fixed-base exponentiation with multiplicative splitting with *R* prime

In this section we present a novel recoding algorithm based on multiplicative splitting modulo R prime. We will show that the resulting exponentiation algorithm can be made constant time (Table 3).

4.1 Digit multiplicative splitting for prime radix *R*

We present in this subsection a variant of the multiplicative splitting to the case of a prime radix R. When R is a prime, we can use a multiplicative splitting modulo R based on an extension of the half-size multiplicative splitting of [18]. Our goal is to get the following splitting

$$k_i = k_i^{(0)}(k_i^{(1)})^{-1} \mod R \text{ with } \begin{cases} |k_i^{(0)}| < c \\ |k_i^{(1)}| \leq R/c \end{cases} \tag{8}$$

for a fixed bound $0 < c < R$.

4.1.1 Multiplicative splitting modulo a prime *R*

The multiplicative splitting modulo a prime radix R is based on the extended Euclidean algorithm. We briefly review this algorithm. We consider a prime integer R and $0 < k < R$. Then k and R are pairwise prime $\gcd(k, R) = 1$. The Euclidean algorithm computes $\gcd(k, R)$ through a sequence of modular reductions:

$$r_0 = R, r_1 = k, r_2 = r_0 \bmod r_1, \ldots$$
$$\ldots, r_{j+1} = r_{j-1} \bmod r_j, \ldots$$

The sequence of remainders r_j satisfies

$$\gcd(r_j, r_{j+1}) = \gcd(R, k)$$

and is strictly decreasing and thus reaches 0 after some iterations. The last $r_\ell \neq 0$ satisfies $r_\ell = \gcd(k, R) = 1$. The extended Euclidean algorithm computes a Bezout relation

$$uR + vk = \gcd(k, R)$$

by maintaining two sequences of integers u_j and v_j satisfying:

$$u_j R + v_j k = r_j, \text{ for } j = 0, 1, \ldots, \ell. \tag{9}$$

The sequence v_j is an increasing sequence in magnitude starting from $v_0 = 0$ and $v_1 = 1$. The multiplicative splitting of (8) can then be obtained from (9) where we take j such that $r_j \in [0, c[$ and $v_j \in [0, R/c]$ and by taking $k_i^{(0)} = r_j$ and $k_i^{(1)} = v_j$. The following lemma establishes this property.

Lemma 1 *If one chooses $c \in [0, R[$, there exists j such that $|r_j| \geq c$ and $r_{j+1} < c$ and at the same time $|v_j| \leq R/c$ and $|v_{j+1}| \geq R/c$.*

The proof of the lemma is given in the appendix.

This leads to the method shown in Algorithm 7 for multiplicative splitting modulo a prime radix R. In this algorithm, a third variable s is used for the sign of the multiplicative splitting.

Table 3 Complexity of exponentiation based on multiplicative splitting recoding with $R = m_0 m_1$

Complexity		
Step	Operation	Cost
$1\times$ Step 6	$T[i][k_i'^{(0)}] \times T[i][-1]$	1 M
$(\ell/m_1 - 1)\times$ Step 8	$Y_0 \times T[i][k_i'^{(0)}] \times T[i][-1]$	2 M
$(m_1 - 1)\times$ Step 11	–	–
$(\ell\frac{m_1-1}{m_1} - (m_1 - 1)) \times$ Step 13	$Y_{k_i'^{(1)}} \times T[i][k_i'^{(0)}]$	1 M
$1\times$ Step 15	$Y_{\lvert k_\ell' \rvert} \times T[\ell][-1]$	1 M
$(W - 1)\times$ Step 18	$X \leftarrow X^2$	1 S
$(\mathcal{H} - 1)\times$ Step 21	$X \times Y_j$	1 M
$1\times$ Step 22	$(X \times Y_0)$	1 M
Total	$(\ell\frac{m_1+1}{m_1} - m_1 + \mathcal{H} + 1)$ M $+(W - 1)$ S	
Total storage	$(m_0 + 1) \times \ell + 1$ elements of G	

Algorithm 7 Truncated Extended Euclidean Algorithm (TruncatedEEA(k, R, c))

Require: $k \in \mathbb{Z}$, the prime radix R, and c, the upper bound for $k_i^{(1)}$.
Ensure: $(s, k^{(0)}, k^{(1)})$, such as $k = |s \times k^{(0)} \times (k^{(1)})^{-1}|_R$ with $0 \leq k^{(0)} < c$ and $0 \leq k^{(1)} \leq \lceil R/c \rceil$ and $s \in \{-1, 1\}$ when $\gcd(k, R) = 1$.
1: **if** $\gcd(k, R) = R$ **then**
2: **return** $(1, 0, 0)$
3: **else**
4: $u_0 \leftarrow 1, v_0 \leftarrow 0, r_0 \leftarrow R, u_1 \leftarrow 0, v_1 \leftarrow 1, r_1 \leftarrow |k|_R$
5: **while** $(r_1 \geq c)$ **do**
6: $q \leftarrow \lfloor r_0/r_1 \rfloor, r_2 \leftarrow |r_0|_{r_1}$
7: $u_2 \leftarrow u_0 - q \cdot u_1, v_2 \leftarrow v_0 - q \cdot v_1$
8: $(u_0, v_0, r_0) \leftarrow (u_1, v_1, r_1)$
9: $(u_1, v_1, r_1) \leftarrow (u_2, v_2, r_2)$
10: $s \leftarrow \text{sign}(v_1), k^{(0)} \leftarrow r_1, k^{(1)} \leftarrow |v_1|$
11: **return** $(s, k^{(0)}, k^{(1)})$

4.1.2 Recoding the exponent

We now present our recoding approach for an integer k given in radix-R representation:

$$k = \sum_{i=0}^{\ell-1} k_i R^i, \text{ with } \ell = \lceil t/\log_2(R) \rceil.$$

We choose a splitting bound c and we consider a digit $k_i \neq 0$. Using Algorithm 7 we get $s_i, k_i^{(0)}$ and $k_i^{(1)}$ such that

$$k_i = s_i k_i^{(0)} (k_i^{(1)})^{-1} \mod R \text{ with } \begin{cases} s_i \in \{-1, 1\} \\ k_i^{(0)} \in [0, c[, \\ k_i^{(1)} \in [0, R/c]. \end{cases} \quad (10)$$

We put apart the case $k_i = 0$ which is recoded as $(1, 0, 0)$ (cf. Step 2 of Algorithm 7). We handle the reduction modulo R as follows:

$$C = (s_i k_i^{(0)} |(k_i^{(1)})^{-1}|_R - k_i)/R \text{ (exact quotient)},$$
$$k_i = s_i k_i^{(0)} |(k_i^{(1)})^{-1}|_R - CR.$$

One notices that C satisfies $-c \leq C < c$. We then consider C as a carry that we subtract to k_{i+1}.

We obtain an expression $k = \sum_{i=0}^{\ell} k'_i R^i$ of k in radix R such that each digit $k'_i = s_i k_i^{(0)} |(k_i^{(1)})^{-1}|_R$ is given in a multiplicative splitting form. The last coefficient $k'_\ell = -C$ is necessary to handle the last carry. The resulting recoding algorithm is shown in Algorithm 8.

Example 3 We present an example of multiplicative splitting recoding for a prime radix $R = 89$ with an exponent size t of 20 bits ($0 < k < 2^{20}$). In this case, one has $\ell = \lceil 20/\log_2(89) \rceil = 4$. One also sets $c = 2^3 = 8$, and

Algorithm 8 Multiplicative Splitting Recoding for R Prime

Require: R prime, $k = \sum_{i=0}^{\ell-1} k_i R^i$, and c the splitting bound.
Ensure: $\{(s_i, k_i^{(0)}, k_i^{(1)}), 0 \leq i < \ell, (k'_\ell)\}$ the multiplicative splitting recoding of k.
1: $C \leftarrow 0$
2: **for** i **from** 0 **to** $\ell - 1$ **do**
3: $k_i \leftarrow k_i - C$
4: $s_i, k_i^{(0)}, k_i^{(1)} \leftarrow \text{TruncatedEEA}(k_i, R, c)$.
5: $C \leftarrow (s_i k_i^{(0)} |(k_i^{(1)})^{-1}|_R - k_i)/R$ //exact quotient
6: **return** $\{(s_i, k_i^{(0)}, k_i^{(1)}), 0 \leq i < \ell, (k'_\ell = -C)\}$

then, $\lceil R/c \rceil = 12$. Let us take $k = 901644_{10}$, the random exponent. By rewriting k in radix-R, one has

$$k = 74 + 73 \cdot 89 + 24 \cdot 89^2 + 1 \cdot 89^3.$$

The execution trace of Algorithm 8 is provided in Table 4.

4.2 Exponentiation algorithm with multiplicative splitting recoding in a prime radix *R*

We now present an exponentiation algorithm which takes advantage of the exponent recoding given in Sect. 4.1.2. One wants to compute

$$g^k = g^{\sum_{i=0}^{\ell} k'_i \cdot R^i} = g^{k'_\ell \cdot R^\ell} \cdot \prod_{i=0}^{\ell-1} g^{k'_i \cdot R^i} \quad (11)$$

with

$$g^{k'_i \cdot R^i} = g^{s_i \cdot k_i^{(0)} \cdot |(k_i^{(1)})^{-1}|_R \cdot R^i}, \text{ if } k_i^{(1)} \neq 0,$$
$$g^{k'_i \cdot R^i} = 1, \text{ if } k_i^{(1)} = 0 \text{(this corresponds to } k_i = 0).$$

Table 4 Example of an execution trace of Algorithm 8

Iter.	Step	Value
$i = 0$	3:	$k_0 = 74$ does not change since $C = 0$
	4:	$s_0 = -1, k_0^{(0)} = 1, k_0^{(1)} = 6$.
	5:	$C \leftarrow (s_0 \cdot k_0^{(0)} \cdot |(k_0^{(1)})^{-1}|_R - k_0)/R = -1$
$i = 1$	3:	$k_1 \leftarrow 73 + 1 = 74$ since $C = -1$
	4:	$s_1 = -1, k_1^{(0)} = 1, k_1^{(1)} = 6$.
	5:	$C \leftarrow (s_1 \cdot k_1^{(0)} \cdot |(k_1^{(1)})^{-1}|_R - k_1)/R = -1$
$i = 2$	3:	$k_2 \leftarrow 24 + 1 = 25$ since $C = -1$
	4:	$s_2 = -1, k_2^{(0)} = 3, k_2^{(1)} = 7$.
	5:	$C \leftarrow (s_2 \cdot k_2^{(0)} \cdot |(k_2^{(1)})^{-1}|_R - k_2)/R = -2$
$i = 3$	3:	$k_3 \leftarrow 1 + 2 = 3$ since $C = -2$
	4:	$s_3 = 1, k_3^{(0)} = 3, k_3^{(1)} = 1$.
	5:	$C \leftarrow (s_3 \cdot k_3^{(0)} \cdot |(k_3^{(1)})^{-1}|_R - k_3)/R = 0$
$((-1, 1, 6), (-1, 1, 6), (-1, 3, 7), (1, 3, 1))$ and $k'_4 = C = 0$		

In order to compute the Fixed-base exponentiation $g^k \bmod p$, one stores the following values:

$$T[i][s][j] = g^{R^i \cdot s \cdot |j^{-1}|_R}, \text{ with } \begin{cases} 0 \le i \le \ell - 1, \\ 1 \le j \le \lceil R/c \rceil, \\ s \in \{-1, 1\}. \end{cases}$$
$$T[i][s][0] = 1 \text{ with } s \in \{-1, 1\}.$$
$$T[\ell][s] = g^{sR^\ell} \text{ with } s \in \{-1, 1\}.$$

One denotes Y_j the product of the terms $g^{s_i \cdot |(k_i^{(1)})^{-1}|_R \cdot R^i}$ such that of $k_i^{(0)} = j$. This means that for $j \neq |k'_\ell|$

$$Y_j = \left(\prod_{k_i^{(0)} = j} T[i][s_i][k_i^{(1)}] \right).$$

and for $j = |k'_\ell|$ one has

$$Y_j = \left(\prod_{k_i^{(0)} = j} T[i][s_i][k_i^{(1)}] \right) \times T[\ell][\text{sign}(k'_\ell)].$$

We can then rewrite the products in (11) in terms of Y_j as follows:

$$g^k = \prod_{j \in \{1, \ldots, c-1\}} Y_j^j.$$

Every individual exponentiation Y_j^j is performed with a square-and-multiply approach, which is more efficient than performing $j - 1$ multiplications, even for small c. This finally leads to the exponentiation shown in Algorithm 9.

The above algorithm can be implemented in a constant-time fashion. Indeed there is no **if** control attached to the digits of the exponent. Then, the algorithm consists in a constant and regular sequence of multiplications and squarings as soon as a multiplication with a 1 is computed as any other multiplication.

Example 4 We consider the exponent $k = 901644_{10}$ along with the multiplicative splitting recoding computed in Example 3.

$$((-1, 1, 6), (-1, 1, 6), (-1, 3, 7), (1, 3, 1)) \text{ and } k'_4 = 0. \tag{12}$$

We present the computation of g^k using Algorithm 9. In terms of storage, one computes the values

$$T[i][s][j] = g^{R^i \cdot s \cdot |j^{-1}|_R} \text{ with } \begin{cases} 0 \le i \le \ell - 1, \\ 1 \le j \le \lceil R/c \rceil = 12, \\ s \in \{-1, 1\}. \end{cases}$$

Algorithm 9 Fixed-base exponentiation with multiplicative splitting for prime radix R

Require: R a prime integer, an exponent $k = \sum_{i=0}^{\ell-1} k_i R^i$ and $\{(s_i, k_i^{(0)}, k_i^{(1)}), 0 \le i < \ell, k'_\ell\}$ the multiplicative splitting recoding in radix R of k and $g \in G$.
Ensure: $X = g^k$
1: *Offline precomputation.* For $0 \le i \le \ell - 1, 1 \le j \le \lceil R/c \rceil, s \in \{-1, 1\}$ store $T[i][s][j] \leftarrow g^{R^i \cdot s \cdot |j^{-1}|_R}$ and $T[i][s][0] \leftarrow 1$ for $0 \le i \le \ell - 1, s \in \{-1, 1\}$ and $T[\ell][s] \leftarrow g^{sR^\ell}$ for $s \in \{-1, 1\}$.
2: $X \leftarrow 1, Y_j \leftarrow 1$ for $0 \le j \le c$
3: **for** i **from** 0 **to** $\ell - 1$ **do**
4: $\quad Y_{k_i^{(0)}} \leftarrow Y_{k_i^{(0)}} \times T[i][s_i][k_i^{(1)}]$
5: $Y_{|k'_\ell|} \leftarrow Y_{|k'_\ell|} \times T[\ell][\text{sign}(k'_\ell)]$
6: $W \leftarrow$ size of c in bits
7: **for** i **from** $W - 1$ **downto** 0 **do**
8: $\quad X \leftarrow X^2$
9: $\quad$ **for** j **from** $c - 1$ **downto** 1 **do**
10: $\quad\quad$ **if** bit i of j is non zero **then**
11: $\quad\quad\quad X \leftarrow X \times Y_j$
12: **return** (X)

One has the following values of $\left|j^{-1}\right|_R$ for $1 \le j \le 12$

$$\{1, 45, 30, 67, 18, 15, 51, 78, 10, 9, 81, 52\}.$$

This brings us to store the following values in G:

$$T[i][1] = \{g^{89^i}, g^{89^i \cdot 45}, g^{89^i \cdot 30}, g^{89^i \cdot 67}, g^{89^i \cdot 18}, g^{89^i \cdot 15}, g^{89^i \cdot 51}, g^{89^i \cdot 78}, g^{89^i \cdot 10}, g^{89^i \cdot 9}, g^{89^i \cdot 81}, g^{89^i \cdot 52}\}$$
$$T[i][-1] = \{g^{-89^i}, g^{-89^i \cdot 45}, g^{-89^i \cdot 30}, g^{-89^i \cdot 67}, g^{-89^i \cdot 18}, g^{-89^i \cdot 15}, g^{-89^i \cdot 51}, g^{-89^i \cdot 78}, g^{-89^i \cdot 10}, g^{-89^i \cdot 9}, g^{-89^i \cdot 81}, g^{-89^i \cdot 52}\}.$$

The execution of Algorithm 9 is shown step by step in Table 5

4.3 Complexity

Let us now evaluate the complexity of Algorithm 9. Concerning the amount of storage it consists in $2(\lceil R/c \rceil \times +1)\ell + 2$ elements of G.

For the online complexity, we evaluate the cost of each step of Algorithm 9 based on the following:

- the multiplications in Step 4 are performed even in case of $Y_{k_i^{(0)}} = 1$, in order to ensure the constant time of the computation;
- the same applies for Step 5.

The number of operations in the final reconstruction is evaluated as follows:

- the squaring in Step 8 is not performed in the first loop iteration ($X = 1$);

Table 5 Example of an execution trace for an exponentiation based on multiplicative splitting recoding with R prime

Iter.	Step	Coeff	Value
		$s_0 = -1$	
$i = 0$	4:	$k_0^{(0)} = 1$ $k_0^{(0)} = 6$	$Y_1 \leftarrow Y_1 \times T[0][s_0][k_0^{(1)}] = 1 \times g^{-15}$
		$s_1 = -1$	
$i = 1$	4:	$k_1^{(0)} = 1$ $k_1^{(1)} = 6$	$Y_1 \leftarrow Y_1 \times T[1][s_1][k_1^{(1)}] = g^{-15} \times g^{-89 \cdot 15} = g^{-1350}$
		$s_2 = -1$	
$i = 2$	4:	$k_2^{(0)} = 3$ $k_2^{(1)} = 7$	$Y_3 \leftarrow Y_3 \times T[2][s_3][k_2^{(1)}] = 1 \times g^{-89^2 \cdot 51} = g^{-403971}$
		$s_3 = 1$	
$i = 3$	4:	$k_3^{(0)} = 3$ $k_3^{(1)} = 1$	$Y_3 \leftarrow Y_3 \times T[3][s_3][k_3^{(1)}] = g^{-403971} \times g^{89^3 \cdot 1} = g^{300998}$
–	5:	$k'_4 = 0$	$Y_0 \leftarrow Y_0 \times T[\ell][\text{sign}(k'_4)] = g^{59969536}$
–	7: to 11:	–	$g^k = \prod_{j=1}^{c-1} Y_j^j = g^{3 \cdot 300998 - 1350} = g^{901644}$

Table 6 Exponentiation complexity and storage for the proposed approach with a prime radix R recoding

Complexity		
Step	Operation	Complexity
$\ell \times$ Step 4	$Y_{k_i^{(0)}} \times T[i][s_i][k_i^{(1)}]$	1 M
$1 \times$ Step 5	$Y_{\lvert k'_\ell \rvert} \times T[\ell][\text{sign}(k'_\ell)]$	1 M
$(W-1) \times$ Step 12	X^2	1 S
$(\mathcal{H}-1) \times$ Step 15	$X \times Y_j$	1 M
Total	$(\ell + \mathcal{H})$ M $+(W-1)$ S	
Total storage	$2(\lceil R/c \rceil + 1)\ell + 2$ elements of G	

– This first multiplication in Step 11 is skipped since it is an affectation. The other multiplications in Step 11 are performed even in case of $Y_j = 1$, again to ensure a constant computation time.

We denote by $\mathcal{H}$ the sum of the j Hamming weights for each j from $c-1$ downto 1 (`for` loop in Step 7). The value of $\mathcal{H}$ is as follows for the different values of c can be found in Table 2.

The contribution of each step is given in Table 6 along with the total complexity.

5 Complexity and experimentation comparison

5.1 Complexity comparison

In Table 7 we give the complexities in terms of the number of online operations and storage amount of the state-of-the-art approaches (Sect. 2) and the two proposed approaches in Sects. 3 and 4. All the approaches presented in the above table can be implemented in constant time except the Square-and-multiply, Fixed-base NAF$_w$ and the proposed approach with $R = m_0 m_1$.

Let us first see when the Fixed-base Comb method is better than the Fixed-base Radix-R exponentiation. We denote w_C the window size of the Comb method and w_R the one of the Radix-R method. In order to have both methods with the same number of online operations in G, we take $w_C = 2w_R$: in this case, both methods require t/w_R online operations in G. Then, considering the storage amount when $w_C = 2w_R$, one can see that the Comb method requires 2^{2w_R} while the Radix-R method needs $\frac{t}{w_R} 2^{w_R}$ elements of G. In other words, for a fixed number t/w_R of online computation, the Comb method is better than the Radix-R as soon as $2^{w_R} < \frac{t}{w_R}$ which is the case for small w_R, i.e., for small amount of storage.

If we now consider the Fixed-base NAF$_w$, we can notice that it does not compare favorably with the radix-R approach. Indeed for $e = 1$ we would have almost the same number of online multiplications, whereas the amount of data in the NAF$_w$ is larger by a factor of w. For larger value of e the number of squarings would increase quickly rendering the approach not competitive. Moreover the Fixed-base NAF$_w$ has the major drawback to not be constant time.

It is more difficult to formally compare the proposed approaches with the Comb and Radix-R approaches. Indeed, they involve a third parameter (c or m_1), which means that for a fixed number of online operations, we would have to find the proper parameter which minimizes the amount of storage. We can still notice that for a given c (resp. m_1) we divide by c (resp. m_1) the amount of storage compared to the Radix-R approach while having an increase in online compu-

Table 7 Complexities and storage amounts of exponentiation algorithm, average case, binary exponent length t

	Constant time	#Mul	#Squ.	Storage (#values in G)
Square-and-mult. (Algorithm 1)	No	$\frac{t}{2}$	$t-1$	0
Fixed-base Radix-$R^{(*)}$ (Algorithm 2)	Yes	$\frac{t}{w}-1$	0	$\frac{t}{w}2^w$
Fixed-base Comb (Algorithm 3)	Yes	$\frac{t}{w}-1$	$\frac{t}{w}-1$	2^w
Fixed-base NAF$_w$ (Algorithm 3)	No	$\frac{t}{w}(1-\left(\frac{w}{w+1}\right)^w)$	$(e-1)w$	$\frac{t}{e}2^{w-2}$
Proposed$^{(*)}$ with $R=m_0m_1$ (Algorithm 6)	No	$\frac{t}{w}\frac{m_1+1}{m_1}-m_1+\mathcal{H}+1$	$W-1$	$(2^w/m_1+1)\frac{t}{w}+1$
Proposed$^{(*)}$ with R prime (Algorithm 9)	Yes	$\frac{t}{w}+\mathcal{H}$	$W-1$	$(2^{w+1}/c+1)\frac{t}{w}+1$

$^{(*)}$ We assume that R is a w bit integer

tation ($\mathcal{H}$ and W). This means that the proposed approaches can be competitive only for small c and m_1.

To have a clearer idea of the impact of the proposed approach so we follow the strategy used in [16]. Indeed, for practical sizes of group and exponent and for different level of online operations, we evaluate the best choice of parameters which minimizes the amount of precomputation. In the sequel we give the results for DSA and ECDSA, for the fields and curves recommended by the NIST.

5.2 Complexities and timings for modular exponentiation

In this subsection we focus on exponentiation in $((\mathbb{Z}/p\mathbb{Z})^*, \times)$ used in DSA. We evaluate and compare the complexities of the best method of the literature, i.e., Fixed-base Comb (Algorithm 3) and Fixed-base Radix-R (Algorithm 2), with the complexity of our proposed approaches based on a multiplicative splitting recoding of the exponent (Algorithm 6 for $R = m_0m_1$ and Algorithm 9 for R prime).

In the sequel of this subsection, we provide complexity evaluations in terms of modular multiplications MM, under the assumption of modular squaring MS = 0.86 MM, which is the average value of our implementations for the NIST DSA recommended field sizes. We warn the reader to keep in mind that the Fixed-base Comb, Radix-R and Algorithm 9 are constant time, and that Algorithm 6 is not, i.e., the only one weak against timing attacks.

The NIST provides recommended key sizes and corresponding field sizes (respectively, the size of the primes q and p, see NIST SP800-57 [2]). This standardized sizes are as follows:

Figure 1 gives the general behavior of the four algorithms in terms of storage (y axis) with respect to the number of online operations (x axis). In the figure, we present three of the field sizes recommended in the NIST standards (see [2]) and the behavior is roughly the same for all sizes, although the benefit of our approach with $R = m_0m_1$ is lower for smaller sizes. One can see that the Fixed-base Comb method is the best for small storage amount. Our m_0m_1 approach (Algorithm 6) is better for larger amount of storage; however, the Fixed-base Radix-R method is the best when the storage is increasing. One can see that the R prime multiplicative splitting approach (Algorithm 9) is less efficient than the $R = m_0m_1$ for small storage amounts. The reason is that this requires some additional computations to get a constant time execution, while the m_0m_1 approach is not constant time and is thus slightly more efficient. Nevertheless, one can see a range of storage/complexity trades-off where the R prime multiplicative splitting approach is the best of the constant-time ones (Table 8).

Table 9 shows numerical application of the complexity comparison between the Fixed-base Comb (Algorithm 3), the Fixed-base Radix-R (Algorithm 2) and the approaches based on our multiplicative splitting recodings (Algorithm 6 and Algorithm 9). For an equivalent number of MMs, we provide the minimum amount of storage. We can notice the following:

- For all key sizes, we do not provide the results for small amount of storage (values for $w < 8$). For such storage, the Fixed-base Comb method is the best. One may notice that the Fixed-base Radix-R approach involves the largest storage amount at this complexity level.
- *Comparison of the two proposed approaches: $R = m_0m_1$ vs R prime.* We would like to evaluate the improvements provided by the new approach (Algorithm 9) compared to (Algorithm 6) which was presented at WAIFI 2016. The results in Table 9 show that the exponentiation with multiplicative splitting with $R = m_0m_1$ and R prime are close from each other. But the approach with $R = m_0m_1$ is generally slightly better than the one with R prime. But, as noticed earlier, this is the price to pay to get a constant-time algorithm.
- *Comparison of constant-time approaches.* We consider the Fixed-base Comb, Radix-R and multiplicative splitting with R prime approaches. A thorough analysis of

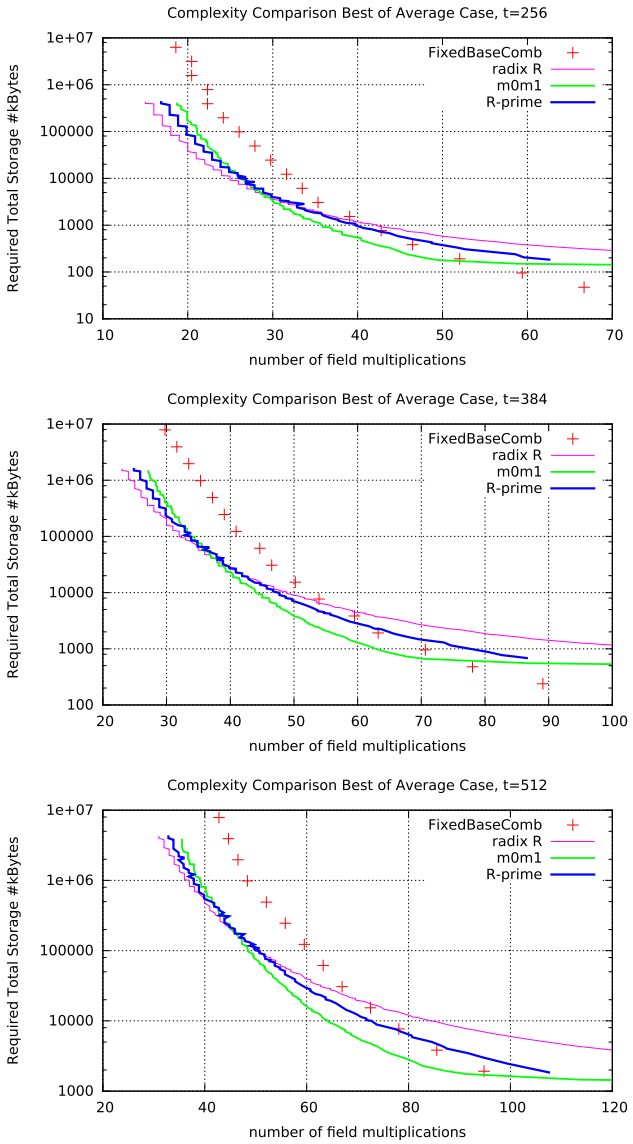


Fig. 1 Complexity comparison, Fixed-base modular exponentiation NIST DSA, key size 256, 384 and 512 bits (field size 3072, 7360 and 15,360 bits)

the complexities shows that the proposed approach is interesting for intermediate level of online computation. Specifically from Table 9, for a 224 bit key size, one notices that there are not many cases where the proposed multiplicative splitting approach is interesting. However, for the other key sizes $t = 256$, 384 and 512, one can see a lot of cases where the amount of storage is reduced by 50% compared to Comb and Radix-R approaches.

Remark 1 One may notice that the largest memory storage sizes exceed the common values of random-access memory, and in some cases, the maximum allowed for the `malloc` function of the standard C library for memory allocation.

Table 8 NIST recommended key and field sizes

Security level	80	112	128	192	256
Key size (bits)	160	224	256	384	512
Field size (bits)	1024	2048	3072	7680	15,360

Nevertheless, the storage savings proposed by our method and Fixed-base Radix-R ones allow to keep the level of storage under the limit for lower complexities.

5.2.1 Implementation results

Implementation strategies We review hereafter the main implementation strategies and test process for modular exponentiation for NIST recommended sizes. This applies for the four considered exponentiation algorithms. The algorithms were coded in C, compiled with `gcc 4.8.3` and run on the same platform.

- *Multi-precision multiplication and squaring* We used the low level functions performing multi-precision multiplication and squaring of the `GMP` library as building blocks of our codes (`GMP 6.0.0`, see GMP library [20]). According to the GMP documentation, the classical schoolbook algorithm is used for small sizes, and Karatsuba and Toom-Cook subquadratic methods for size $\geq$ 2048 bits.
- *Modular reduction* This operation implements the Montgomery representation and modular reduction method, which avoid multi-precision division in the computation of the modular reduction. This approach was presented by Montgomery in [17]. We use the block Montgomery algorithm suggested by Bosselaers et al. [3]. In this algorithm, the multi-precision operations combine full size operand with one word operand and are also available in the GMP library [20].
- *Multiplicative splitting recoding with $R = m_0 m_1$ and R prime* The conversion in radix-R needs multi-precision divisions. These operations are implemented using the GMP library [20]. The size of these operations is decreasing along the algorithm, and this is managed through GMP. The other operations are classical long integer operations. At Step 11 in Algorithm 5 (resp. Step 5 in Algorithm 8), an inversion modulo m_0 (resp. R) is required. This operation is performed using the Extended Euclidean Algorithm, over long integer data. For the considered exponent sizes, the cost of the recoding is negligible. This is explained by the small size of the exponent in comparison with the size of the data processed during the modular exponentiation (see Table 8). The timings given in the next subsection include this recoding.

Table 9 Storage amount comparison for Fixed-base Comb, Fixed-base Radix-R and modular exponentiation with multiplicative splitting recoding for NIST recommended exponent sizes

#MM	Fixed-base Comb	Fixed-base Radix-R	Multiplicative splitting	
			$R = m_0 m_1$	R-prime
Key size $t = 224$ *bits*				
45	127.5 kB	345 kB	**108** kB	240 kB
	$w = 9$	$R = 31$	$(m_0, m_1) =$	$(R, c) =$
			(11, 9)	(97, 7)
37	511.5 kB	594 kB	**242** kB	541 kB
	$w = 11$	$R = 61$	(31, 7)	(179, 5)
30	4095.5 kB	1386 kB	**770** kB	**1205** kB
	$w = 14$	$R = 179$	(127, 7)	(179, 5)
24	32,767.5 kB	4230 kB	**4173** kB	4489 kB
	$w = 17$	$R = 677$	(877, 7)	(1223, 3)
19	524,287.5 kB	27,084 kB	50,409 kB	27,954 kB
	$w = 21$	$R = 5417$	(13,441, 5)	(6211, 2)
Key size $t = 256$ *bits*				
46	383 kB	845 kB	**241** kB	494 kB
	$w = 10$	$R = 47$	$(m_0, m_1) =$	$(R, c) =$
			(17, 11)	(97, 5)
39	1535 kB	1454 kB	**579** kB	**1116** kB
	$w = 12$	$R = 97$	47; 7	223; 5
32	12,287 kB	3179 kB	**2070** kB	**3084** kB
	$w = 15$	$R = 257$	211; 6	409; 3
26	98,303 kB	9846 kB	**9642** kB	10207 kB
	$w = 18$	$R = 937$	1223; 6	1699; 3
20	15,728,63 kB	**66**,**676** kB	225,482 kB	85,558 kB
	$w = 22$	$R = 8467$	37,579; 5	12,007; 2
Key size $t = 384$ *bits*				
63	1918 kB	4081 kB	**969** kB	2274 kB
	$w = 11$	$R = 67$	$(m_0, m_1) =$	$(R, c) =$
			(19, 11)	(127, 6)
50	15,358 kB	10,087 kB	**3742** kB	**7182** kB
	$w = 14$	$R = 191$	101; 11	433; 5
41	122,878 kB	26655 kB	**17**,**284** kB	**22**,**891** kB
	$w = 17$	$R = 677$	541; 6	937; 3
35	983,038 kB	80,357 kB	**647**,**68** kB	**65**,**837** kB
	$w = 20$	$R = 2381$	2381; 6	3191; 3
30	78,643,18 kB	246,070 kB	315,053 kB	**235**,**255** kB
	$w = 23$	$R = 8467$	13,441; 5	13,441; 3
26	62,914,558 kB	**951**,**217** kB	32,562,78 kB	10,306,42 kB
	$w = 26$	$R = 37,579$	165,397; 5	43,973; 2
24	50,331,647,8 kB	**17**,**507**,**56** kB	– kB	– kB
	$w = 29$	$R = 74,699$	–	–

Table 9 continued

#MM	Fixed-base Comb	Fixed-base Radix-R	Multiplicative splitting	
			$R = m_0 m_1$	R-prime
Key size $t = 512$ *bits*				
86	3836 kB	9841 kB	**1940** kB	5004 kB
	$w = 11$	$R = 59$	$(m_0, m_1) =$	$(R, c) =$
			(13, 11)	(163, 9)
73	15356 kB	17,855 kB	**4747** kB	**10,005** kB
	$w = 13$	$R = 127$	(41, 10)	(241, 6)
60	122,876 kB	46775 kB	**162,24** kB	**29,979** kB
	$w = 16$	$R = 409$	(179, 11)	(739, 5)
52	491,516 kB	93,110 kB	**54,680** kB	**76,505** kB
	$w = 18$	$R = 937$	(677, 7)	(1223, 3)
48	983,036 kB	156,091 kB	**106,185** kB	**136,971** kB
	$w = 19$	$R = 1699$	(1489, 10)	(2381, 3)
41	78,643,16 kB	489,112 kB	**355,573** kB	**477,551** kB
	$w = 22$	$R = 6211$	(5417, 7)	(6211, 2)
35	62,914,556 kB	20,484,19 kB	21,138,90 kB	**19,499,34** kB
	$w = 25$	$R = 30,347$	(37,579, 7)	(47,269, 3)

Bold values indicate the corresponding storage is better than the one of the state-of-the-art approaches

- *Test processing.* The tests involve a few hundred datasets, which consist of random exponent inputs and an exponentiation base with the precomputed stored values. We compute 2000 times the corresponding exponentiation for each dataset and keep the minimum number of clock cycles. This avoids the cold-cache effect and system issues. The timings are obtained by averaging the timings of all datasets.

Tests results and comparison The four considered exponentiation algorithms were coded in C, compiled with `gcc 4.8.3` and run on the following platform: the CPU is an Intel XEON® E5-2650 (Ivy bridge), and the operating system is CENTOS 7.0.1406. On this platform, the random-access memory is 12.6 GBytes. One notices that the performance results include the Radix-R recoding and the multiplicative splitting of the digits for $R = m_0, m_1$ and R prime.

We show the performance results in Fig. 2 which gives a global overview. The implementation results confirm the complexity evaluation, for key sizes of 224, 256, 384, and 512 bits. However, the best results are for 384 and 512 bits.

In Table 10, we provide the most significant results. The gains shown are roughly in the same order of magnitude as the one of the complexity evaluation. In particular, for the largest key size (512 bits), the storage of our approach with $R = m_0 m_1$ is nearly ten times less than the one required with the Fixed-base Comb method, and nearly 14% less than the one required for the Fixed-base Radix-R method, for the same level of clock cycles. In the same time, our approach with R prime gives equivalent results for low levels of storage, and better results for higher levels of storage.

5.3 Complexities and timings for scalar multiplication

In this subsection, we present complexity results and timings of the Fixed-base scalar multiplication over elliptic curves recommended by NIST.

5.3.1 Complexity comparison

In the Fixed-base elliptic curve scalar multiplication case, the main difference with the modular exponentiation is the negligible cost of the inversion of a group element (i.e., an elliptic curve point). This allows to half the memory requirements, by only storing the points corresponding of the positive sign s_i in the recoded coefficients. We provide in appendix C the version of the scalar multiplication algorithm with multiplicative splitting with R prime which takes advantage of a cheap point subtraction.

When computing the complexities, we noticed that the approach using a multiplicative splitting recoding with $R = m_0 m_1$ was never better than the one with R prime. In addition, the approach with $R = m_0 m_1$ does not provide a constant-time computation. That is why we do not consider the approach with $R = m_0 m_1$ in remainder of this subsection. Specifically, we only deal with constant-time approaches: Fixed-base Comb, Radix-R and multiplicative splitting with R prime.

Table 10 Implementation results for modular exponentiation in terms of clock cycles and storage (kB)

Security level	Level of clock–cycles	Scalar multiplication											
		State-of-the-art methods						Proposed approach					
		Fixed-base Comb			Radix R			$R = m_0 m_1$			R prime		
		Time (#CC)	Storage (kB)	w	Time (#CC)	Storage (kB)	R	Time (#CC)	Storage (kB)	(m_0, m_1)	Time (#CC)	Storage (kB)	(R, c)
112 bits (key 224 bits, field 2048 bits)	220,000	221,108	1023.5	12	227,838	829	91	219,864	580	(89,6)	217,104	1191	(257, 3)
	207,000	210,074	2047.5	13	206,888	1324	163	207,072	766	(127, 7)	206,813	1553	(347, 3)
	148,000	149,690	65,535	18	147,877	7289	1223	146,156	21,599	(5417, 6)	149,490	17,661	(3719, 2)
128 bits (key 256 bits, field 3072 bits)	505,000	524,539	1535	12	502,981	1411	91	501,466	897	(79,6)	509,581	1420	(307, 5)
	450,000	449,397	6143	14	445,871	2251	163	446,444	2056	(211,6)	458,936	2372	(307, 3)
	354,000	356,892	98,303	18	354,640	6414	571	354,071	12,843	(1721, 7)	353,662	15,283	(1699, 2)
192 bits (key 384 bits, field 7680 bits)	4,440,00	44,425,90	1918	11	44,921,91	3430	53	44,095,84	1134	(23, 10)	44,944,71	2171	(127, 6)
	353,000	35,543,39	15,358	14	35,248,96	8290	163	35,514,37	4164	(113, 10)	35,346,20	7100	(433, 5)
	270,000	27,363,41	245,758	18	25,434,80	45,221	1223	27,433,99	29,961	(1031, 7)	27,953,63	31,915	(1381, 3)
256 bits (key 512 bits, field 15,360 bits)	18,600,00	18,632,429	15,536	13	19,260,731	13,765	91	18,550,238	4745	(41, 10)	18,683,547	8653	(257, 7)
	15,000,00	14,848,261	122,876	16	15,401,002	34418	163	14,812,616	22,111	(257, 11)	15,541,482	27,853	(641, 5)
	12,400,00	12,477,816	983,036	19	12,193,232	119061	1223	12,499,600	102,820	(1381, 7)	12,802,926	101,886	(1699, 3)

Test performed on an Intel XEON E5-2650 (Ivy bridge), gcc 4.8.3, CENTOS 7.0.1406

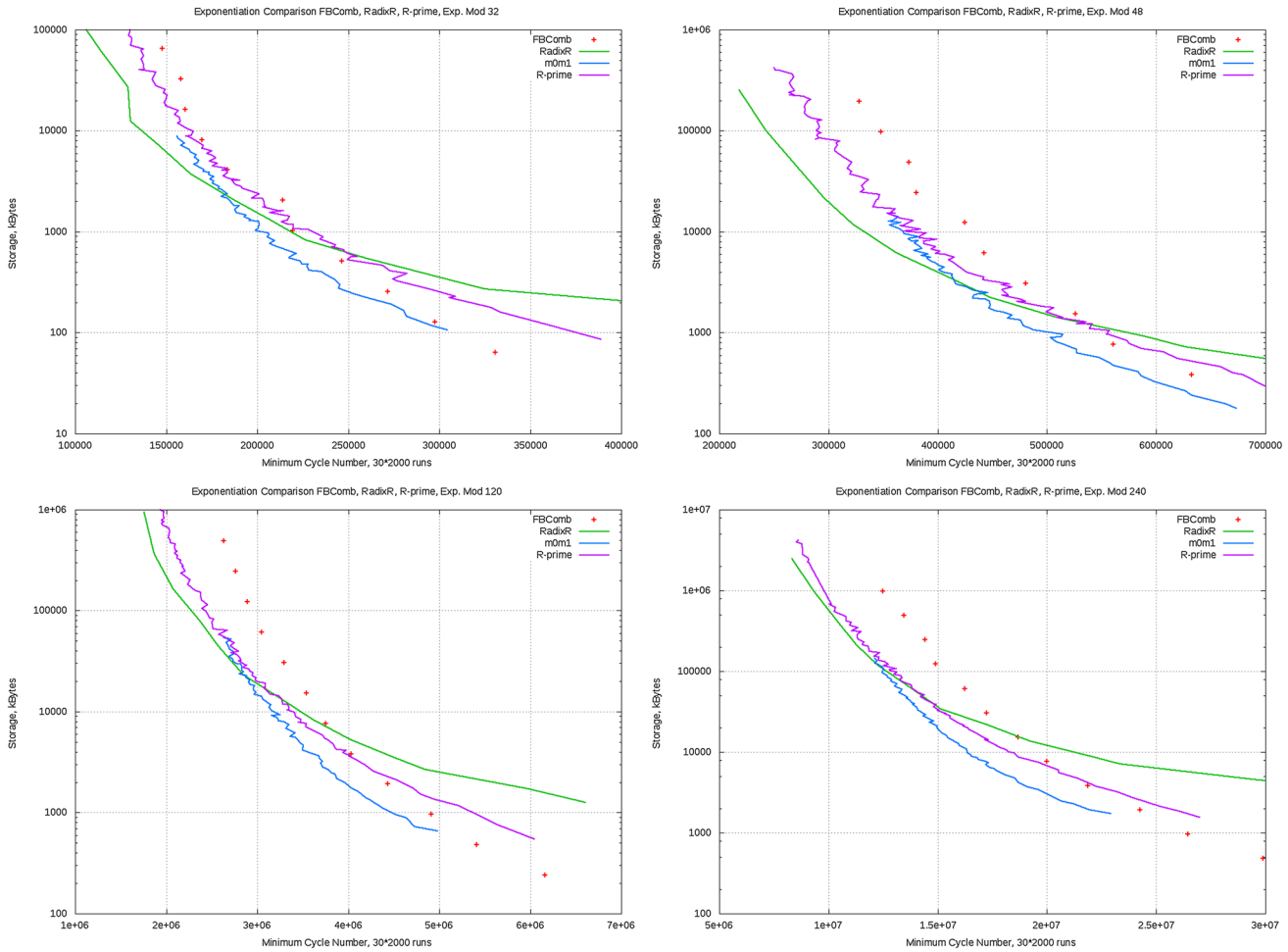


Fig. 2 Performance comparison, Fixed-base modular exponentiation NIST DSA, key size 224, 256, 384 and 512 bits (field size 2048, 3072, 7360 and 15360 bits)

We compare explicit complexities for practical situations, which are the three elliptic curves standardized by NIST: P256, P384, P521. One can find in [11] the Weierstrass curve equations of these three NIST curves which are reviewed in the appendix. For the arithmetic on these curves, we use the Jacobian coordinate system, which provides the fastest curve operations. We use the complexities in terms of operations in $\mathbb{F}_p$ of point addition and doubling for a Weierstrass curves of [6] to derive the complexity of the considered Fixed-base scalar multiplication. The complexities of the curve operations in terms of the number of modular multiplications (MM) and squarings (MS) are as follows:

Addition: $12MM + 4MS$,
Doubling: $4MM + 4MS$,
Mixed-Addition: $7MM + 4MS$.

The resulting complexities of the considered scalar multiplication approaches are reported in Table 11 and Fig. 3 assuming that $MS = 0.8MM$.

Figure 3 gives the general behavior of the storage for a given amount of online computation. This figure shows that, as expected, for small amount of storage the Fixed-base Comb is always the best approach. For larger complexities the proposed approach with a prime radix R is the best choice.

Considering the results in Table 11, one notices that for the four considered methods, one has a slight difference in comparison with the modular exponentiation case. Indeed, for all key sizes and for most of the levels of online computation the proposed approach shows the lowest amount of storage. This is due to the relative cost of the doubling of point and the mixed and full Jacobian addition of points:

- Since the doubling is much cheaper than the additions, this is beneficial to the Fixed-base Comb method and the proposed approach with R prime. Specifically, Fixed-base Comb is the best approach for small amount of

Table 11 Storage amount comparison for constant-time Fixed-base scalar multiplication over NIST curves P256, P384, P521

$\#MM$	Fixed-base Comb	Fixed-base Radix-R	Multiplicative splitting Regular R-prime
NIST curve P256			
441	**64** kB	162 kB	73 kB
	$w = 10$	$R = 59$	$R = 163$; $C = 5$
405	128 kB	222.5 kB	**100** kB
	$w = 11$	$R = 89$	127; 3
317	1024 kB	566 kB	**334** kB
	$w = 14$	$R = 283$	571; 3
264	8192 kB	1522.5 kB	**1142** kB
	$w = 17$	$R = 937$	2381; 3
211	65,536 kB	6235 kB	**5581** kB
	$w = 20$	$R = 4751$	8929; 2
176	10,485,76 kB	22192 kB	**22156** kB
	$w = 24$	$R = 19727$	66467; 3
NIST curve P384			
669	**96** kB	365 kB	127 kB
	$w = 12$	$R = 59$	$R = 149$; $C = 6$
475	1536 kB	1352.5 kB	**661** kB
	$w = 14$	$R = 307$	491; 3
370	24,576 kB	5734 kB	**2901** kB
	$w = 18$	$R = 1699$	2729; 3
299	393,216 kB	26,693 kB	**17,643** kB
	$w = 22$	$R = 9491$	13,441; 2
264	31,457,28 kB	71,250 kB	**51,532** kB
	$w = 25$	$R = 29,231$	43,973; 2
NIST curve P521			
915	**144** kB	678 kB	234 kB
	$w = 10$	$R = 53$	$R = 157$; $C = 7$
651	2304 kB	2547 kB	**1146** kB
	$w = 14$	$R = 283$	739; 5
493	36,864 kB	12,750.5 kB	**6733** kB
	$w = 18$	$R = 1889$	3191; 3
405	589,824 kB	47,627 kB	**35,915** kB
	$w = 22$	$R = 8467$	13,441; 2
352	47,185,92 kB	153,675 kB	**133,905** kB
	$w = 25$	$R = 31,223$	57,709; 2

storage, that is up to several tens of kilobytes, for the three curves P256, P384 and P521. For larger amounts of storage, the other methods remain more efficient.

- The proposed approach which uses a multiplicative splitting recoding with a prime radix R is the best for the following levels of online computations: for P256 and $\#MM \in \{176, ..., 405\}$, for P384 and $\#MM \in \{264, ..., 475\}$ and for P521 and $\#MM \in\in \{352, ..., 651\}$.

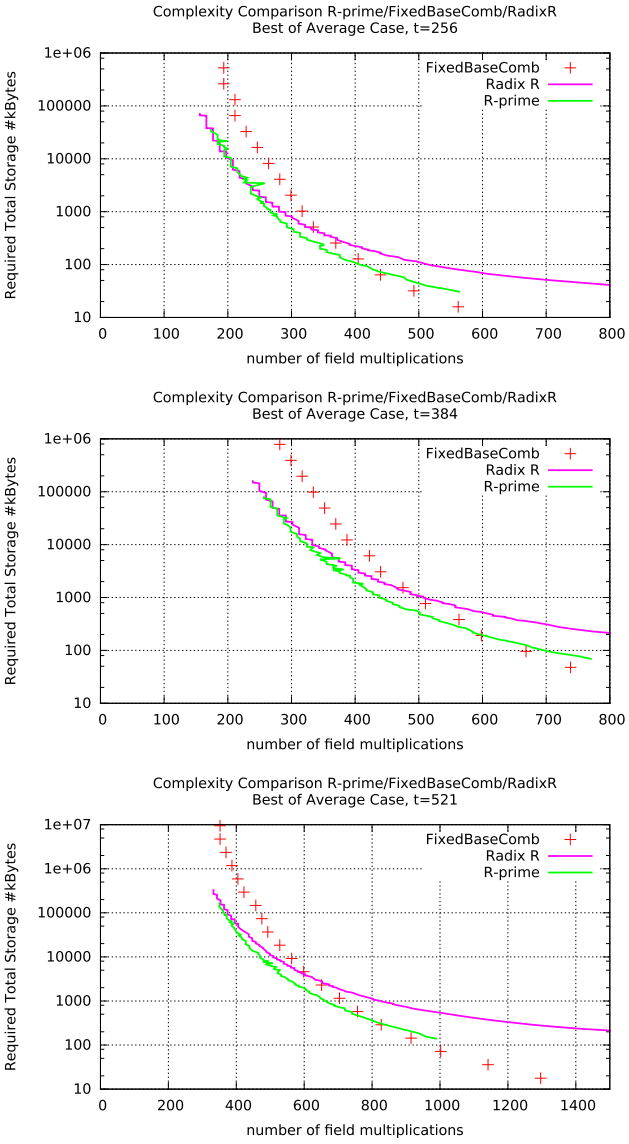


Fig. 3 Complexity comparison for constant-time Fixed-base scalar multiplication on elliptic Weierstrass curves P256, P384 and P521

5.3.2 Implementation results

We now present implementation strategies and results for the constant-time Fixed-base scalar multiplication over NIST prime curves P256, P384, P521.

Implementation strategies The implementation strategies and test processes are the same as the ones presented in Sect. 5.2.1 for modular exponentiation. We review most of them and provide the specific strategies adapted to the considered elliptic curves (Table 12).

- *Curve operations* We use the curve operations in Jacobian coordinate system, which provides the lowest com-

Table 12 Implementation results for Fixed-base scalar multiplication for constant-time algorithms

Security level	Level of Clock cycles	Scalar multiplication								
		State-of-the-art methods						Proposed approach		
		Fixed-base Comb			radix R			R-splitting rec.		
		Time (#CC)	Storage (kB)	w	Time (#CC)	Storage (kB)	R	Time (#CC)	Storage (kB)	(R, c)
128 bits (NIST curve P256)	370,000	378,184	64	12	376,370	74	19	366,057	**37**	(71,5)
	276,000	275,230	1024	14	276,917	231	89	276,660	**170**	(257,3)
	205,000	207,456	32,768	19	206,777	1120	641	203,414	**1012**	(1699,2)
192 bits (NIST curve P384)	575,000	575,854	192	11	571,975	283	41	583,590	**86**	(79,5)
	460,000	461,271	1536	14	470,537	547	97	451,846	**354**	(233,3)
	375,000	376,114	24,576	18	372,952	1861	433	378,733	**1214**	(997,3)
	349,000	359,578	49,151	19	360,786	2069	491	354,919	**1911**	(1699,3)
256 bits (NIST curve P521)	450,000	446,633	288	11	451,280	572	41	449,550	**146**	(97,7)
	364,000	363,615	2304	14	362,166	1621	157	367,299	**726**	(433,5)
	289,000	289,085	73,728	19	288,394	7217	937	290,146	**6243**	(2897,3)

Test performed on an Intel XEON E5-2650 (Ivy bridge), gcc 4.8.3, CENTOS 7.0.1406

Table 13 Recoding tests, for sizes 256, 384 and 521 bits: worst case computation time in clock cycles

Recoding			
Scalar size	256 bits $\#CC$	384 bits $\#CC$	521 bits $\#CC$
Radix R conversion	1200	1640	2250
R-splitting conversion	14,400	21,600	27,500

plexities for the considered NIST curves. The doubling formula is from [1], the mixed addition is from [14] and the full-addition is from [6].

- *Field operations* We use the low level functions performing multi-precision addition, subtraction, multiplication and squaring of the `GMP` library as building blocks of our codes (`GMP 6.0.0`, see GMP library [20]). According to the GMP documentation, the classical schoolbook algorithm is used for small sizes. For the inversion, we use the binary extended Euclidean algorithm, with some specific assembly code portion, which is presented by Brown *et al.* in [4]. The field reductions use also the algorithms presented by Brown et al. [4].
- *Radix R conversion and recoding* The algorithm and the code is the same as the one previously used for modular exponentiation case. However, the size of the scalar in this case is the same as the one of the field elements representing the elliptic curve point coordinates. The computation time of the recoding is no more negligible in this case, as shown by the tests of the conversion alone, and the multiplicative splitting recoding computation (including the conversion). We provide in Table 13 the results of these tests. One sees that the impact of the recoding is at most 8% of the scalar multiplication computation time without recoding, in the worst cases. The most important part of the recoding time is the computation of the multiplicative splittings of the scalar digits, with the repeated use of the Truncated EEA.

  Due to the relative cost of the recoding in the multiplicative splitting with R prime, and to the fact that the recoding itself is not regular as implemented, we provide timings without the recoding, considering that in ECDSA, one can directly generate a random scalar k in a multiplicative splitting form and then process the digital signature (this strategy was proposed in [5] to avoid costly scalar recoding in double base representation).
- *Test processing* The test processing is the same as the one used for the modular exponentiation. This involves a few hundred of datasets, which are random scalars and a Fixed-base with precomputed data. To get the timings, we perform 2000 times the scalar multiplication and keep the minimal number of clock cycles. This is meant to avoid the cold-cache effect and system interruptions. The final timings are the average of the dataset timings.

Test results and comparison The algorithms were coded in C, compiled with `gcc 4.8.3`. The test were performed on a platform which has the following characteristics: an Intel XEON E5-2650 (Ivy bridge), a RAM of 12.6 GBytes and a CENTOS 7.0.1406 operating system.

In Table 12, we report some of the most significant results obtained for the three following approaches: *Fixed-base Comb*, *Fixed-base radix-R* and the one based on multiplicative splitting recoding with a prime radix R. These results show that, except in the last case (P521 and 290,000 clock cycles), our approach provides the smallest storage amount for each considered level of clock cycles. This is consistent with the complexity evaluation shown in Table 11 and Fig. 3. Specifically, for a fixed amount of online computation the proposed approach reduce by roughly 50% the amount of storage.

6 Conclusion

In this paper, we considered Fixed-base modular exponentiation and elliptic curve scalar multiplication. These operations are intensively used in NIST standards for digital signature algorithm. In particular, they are used for remote authentication of web server. We proposed algorithms for modular exponentiation and scalar multiplication based on a multiplicative splitting recoding of the digits of the exponent or scalar. The multiplicative splitting provides more freedom in the distribution of the computational cost into storage and online computation. We evaluated the complexities of the proposed approaches for the security levels recommended by the NIST. We could show that, for a fixed level of computation, the proposed approaches reduce the amount of stored data in most of the considered practical cases. Specifically the storage requirement is reduced by at least 50% in most cases and up to 3 times for the largest NIST standardized key sizes. These complexity results were confirmed by the implementation tests done on an Intel XEON E5-2650.

A: NIST recommended elliptic curves

We review here the NIST recommended curves (see [11]) used in our implementations. Over a finite field $\mathbb{F}_p$, one has:

– Equation of Weierstrass curve:

$$y^2 = x^3 + ax + b$$

with $a = -3$ and $b \in \mathbb{F}_p$.

The NIST curves used :

– **P256:**
One has $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$, that is
```
p = 0xffffffff 00000001 00000000
00000000 00000000 ffffffff ffffffff
ffffffff
```
and
```
b = 0x5ac635d8 aa3a93e7 b3ebbd55
769886bc 651d06b0 cc53b0f6 3bce3c3e
27d2604b
```
Curve subgroup generator:
```
XG = 0x6b17d1f2 e12c4247 f8bce6e5
63a440f2 77037d81 2deb33a0 f4a13945
d898c296
YG = 0x4fe342e2 fe1a7f9b 8ee7eb4a
7c0f9e16 2bce3357 6b315ece cbb64068
37bf51f5
```

– **P384:**
One has $p = 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1$, that is
```
p = 0xffffffff ffffffff ffffffff
ffffffff ffffffff ffffffff ffffffff
fffffffe ffffffff 00000000 00000000
ffffffff
```
and
```
b = 0xb3312fa7 e23ee7e4 988e056b
e3f82d19 181d9c6e fe814112 0314088f
5013875a c656398d 8a2ed19d 2a85c8ed
d3ec2aef
```
Curve subgroup generator:
```
XG = 0xaa87ca22 be8b0537 8eb1c71e
f320ad74 6e1d3b62 8ba79b98 59f741e0
82542a38 5502f25d bf55296c 3a545e38
72760ab7
YG = 0x3617de4a 96262c6f 5d9e98bf
9292dc29 f8f41dbd 289a147c e9da3113
b5f0b8c0 0a60b1ce 1d7e819d 7a431d7c
90ea0e5f
```

– **P521:**
One has $p = 2^{521} - 1$, that is
```
p = 0x01ff ffffffff ffffffff ffffffff
ffffffff ffffffff ffffffff ffffffff
ffffffff ffffffff ffffffff ffffffff
ffffffff ffffffff ffffffff ffffffff
ffffffff
```
and
```
b = 0x0051 953eb961 8e1c9a1f
929a21a0 b68540ee a2da725b 99b315f3
b8b48991 8ef109e1 56193951 ec7e937b
1652c0bd 3bb1bf07 3573df88 3d2c34f1
ef451fd4 6b503f00
```
Curve subgroup generator:
```
XG = 0x00c6 858e06b7 0404e9cd
```

```
9e3ecb66 2395b442 9c648139 053fb521
f828af60 6b4d3dba a14b5e77 efe75928
fe1dc127 a2ffa8de 3348b3c1 856a429b
f97e7e31 c2e5bd66
YG = 0x0118 39296a78 9a3bc004
5c8a5fb4 2c7d1bd9 98f54449 579b4468
17afbd17 273e662c 97ee7299 5ef42640
c550b901 3fad0761 353c7086 a272c240
88be9476 9fd16650
```

B: Proof of Lemma 1

Before proceeding to the proof of Lemma 1 we need to recall the following lemma which states some classical properties of the Extended Euclidean Algorithm. A proof of this lemma can be found in [18].

Lemma 2 *Let v_i and r_i be the two sequences of coefficients computed in Algorithm 7. They satisfy the following properties:*

(i) $(-1)^{i-1}v_i \geq 1$ *for all* $i \geq 1$.
(ii) $v_{i+1}r_i - v_i r_i = (-1)^i R$ *for all* $i \geq 1$.

The proof is a direct consequence of Lemma 2: statements $i)$ and $ii)$ imply that for $i \geq 1$

$$r_{i-1}|v_i| + r_i|v_{i-1}| = R. \tag{13}$$

So if r_{i_c-1} is the last remainder such that $r_{i_c-1} \geq c$ then we have $r_{i_c} < c$. Then taking $i = i_c$ in (13) leads to $r_{i_c-1}|v_{i_c}| + r_{i_c}|v_{i_c-1}| = R$ then one gets $|v_{i_c}| \leq R/r_{i_c-1} \leq R/c$.

C: Fixed-base scalar multiplication based on a multiplicative splitting recoding with prime *R*

We consider an elliptic curve $E(\mathbb{F}_p)$ a point P on the curve and a scalar k. The scalar multiplication based on a multiplicative splitting recoding with prime R is shown in Algorithm 10. Table 14 gives the complexity evaluation.

Algorithm 10 Fixed-base scalar multiplication based on multiplicative splitting recoding with prime radix R

Require: A prime integer R ,a scalar $k = \sum_{i=0}^{\ell-1} k_i R^i$ with $= \{(s_i, k_i^{(0)}, k_i^{(1)}), 0 \leq i < \ell, (k'_\ell)\}$ its multiplicative splitting recoding using W-bit split c and a fixed point $P \in E(\mathbb{F}_p)$.
Ensure: $X = k \cdot P$
1: *Precomputation.*
Store $T[i][j] \leftarrow (\left|j^{-1}\right|_R \cdot R^i) \cdot P$ for $i = 0, \ldots, \ell - 1, j = 1, \ldots, \lceil R/c \rceil$ and $T[\ell][1] \leftarrow R^\ell \cdot P$ and $T[i][0] \leftarrow \mathcal{O}$ for $i = 0, \ldots, \ell - 1$.
2: $X \leftarrow \mathcal{O}, Y_j \leftarrow \mathcal{O}$ for $1 \leq j \leq c$
3: **for** i **from** 0 **to** $\ell - 1$ **do**
4: $\quad Y_{k_i^{(0)}} \leftarrow Y_{k_i^{(0)}} + (s_i) \cdot T[i][k_i^{(1)}]$
5: $Y_{|k'_\ell|} \leftarrow Y_{|k'_\ell|} + (\text{sign}(k'_\ell)) \cdot T[\ell][1]$
6: **for** i **from** W **downto** 0 **do**
7: $\quad X \leftarrow 2 \times X$
8: $\quad$ **for** j **from** $c - 1$ **downto** 1 **do**
9: $\quad\quad$ **if** bit i of j is non zero **then**
10: $\quad\quad\quad X \leftarrow X + Y_j$
11: **return** (X)

Table 14 Complexity evaluation of scalar multiplication based on multiplicative splitting recoding with R prime

Complexity		
Step	Operation	Cost
$\ell\times$ Step 3	$Y_{k_i^{(0)}} + s_i \cdot T[i][k_i^{(1)}]$	$1 MixedAdd$
$1\times$ Step 5	$Y_{\lvert k'_\ell\rvert} + sign(k'_\ell) \cdot T[\ell][1]$	$1 MixedAdd$
$(W-1)\times$ Step 7	$X \leftarrow 2 \times X$	$1 Doubling$
$(\mathcal{H}-1)\times$ Step 10	$X \leftarrow X + Y_j$	$1 Addition$
TOTAL	$(\ell+1) \times MixedAdd$ $+(W-1) \times Doubling + (\mathcal{H}-1) \times Addition$	
Total storage	$\ell \times (\lceil R/c \rceil + 1) + 1$ points on $E(\mathbb{F}_p)$	

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