Recent results on fast multiplication

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Two multiplication problems

Integer multiplication

$$M(n) := \text{cost of multiplying integers with at most } n \text{ digits}$$

- “digits” means in some fixed base (e.g. binary or decimal).
- “cost” means “bit complexity”
  (e.g. # steps on multi-tape Turing machine, or # gates in Boolean circuit).

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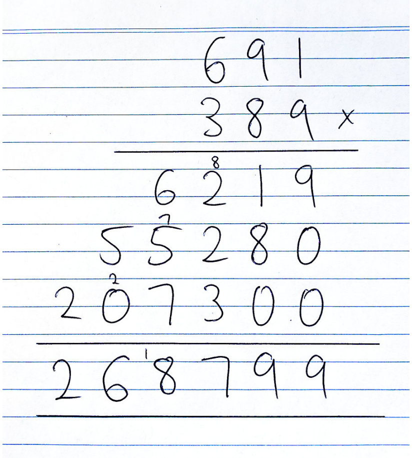
Polynomial multiplication over finite fields

$$M_q(n) := \text{cost of multiplying polynomials in } \mathbb{F}_q[x] \text{ of degree at most } n$$

- $\mathbb{F}_q$ = field with q elements, q a fixed prime power.
- "cost" means bit complexity, or # ring operations in $\mathbb{F}_q$.



Goes back at least to ancient Egypt — probably much older.

Complexity is $M(n) = O(n^2)$.

Same algorithm for polynomials: $M_q(n) = O(n^2)$.

Conjecture (Kolmogorov, around 1956)

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$$M(n) = \Theta(n^2).$$

According to Karatsuba (1995),

> *"Probably, [the conjecture's] appearance is based on the fact that throughout the history of mankind people have been using [the algorithm] whose complexity is $O(n^2)$, and if a more economical method existed, it would have already been found."*

1962	Karatsuba	$n^{\log 3/\log 2}$ ($\approx n^{1.58}$)
1969	Knuth	$n\,2^{\sqrt{2\log n/\log 2}}\log n$
1971	Schönhage–Strassen	$n \log n \log\log n$
2007	Fürer	$n \log n\, K^{\log^* n}$ for some $K > 1$
2019	H.–van der Hoeven†	$n \log n$

Brief history of bounds for $M(n)$.

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Faster multiplication

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Unsolved problem

Can we get $\mathsf{M}_q(n) = O(n \log n)$ unconditionally?

Expected answer: **yes**, because the unproved hypothesis is extremely plausible.

1. Complex DFTs and FFTs
2. Reductions between integer and polynomial multiplication
3. Multidimensional DFTs
4. Conditional $O(n \log n)$ multiplication for integers and polynomials
5. Unconditional $O(n \log n)$ integer multiplication

Complex DFTs and FFTs

The discrete Fourier transform (DFT)

Let $n \geqslant 1$ and $\zeta := e^{2\pi i/n} \in \mathbb{C}$. The roots of $x^n - 1$ are $1, \zeta, \ldots, \zeta^{n-1}$.

The **DFT of length *n*** over $\mathbb{C}$ is the linear map (in fact ring isomorphism)

$$\mathbb{C}[x]/(x^n - 1) \longrightarrow \mathbb{C}^n, \qquad F \longmapsto (F(1), F(\zeta), \ldots, F(\zeta^{n-1})).$$

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Example for $n = 4$

$$\text{The DFT of} \quad F_0 + F_1 x + F_2 x^2 + F_3 x^3 \quad \text{is} \quad \begin{pmatrix} F_0 + F_1 + F_2 + F_3 \\ F_0 + iF_1 - F_2 - iF_3 \\ F_0 - F_1 + F_2 - F_3 \\ F_0 - iF_1 - F_2 + iF_3 \end{pmatrix} \in \mathbb{C}^4.$$

The naive algorithm to evaluate the DFT requires $O(n^2)$ operations in $\mathbb{C}$.

The discrete Fourier transform (DFT)

DFTs can be used to compute **cyclic convolutions**, i.e. multiply in $\mathbb{C}[x]/(x^n - 1)$.

Given as input $F, G \in \mathbb{C}[x]/(x^n - 1)$:

1. use DFT to compute $a_j := F(\zeta^j)$ and $b_j := G(\zeta^j)$ for $j = 0, \ldots, n - 1$
2. compute pointwise products $c_j := a_j \cdot b_j$
3. use inverse DFT to find $H \in \mathbb{C}[x]/(x^n - 1)$ such that $H(\zeta^j) = c_j$ for all j

Output is $H = FG \pmod{x^n - 1}$.

The fast Fourier transform (FFT)

The simplest case of the **Cooley–Tukey FFT** (1965) reduces the complexity of the DFT from $O(n^2)$ to $O(n \log n)$ operations in the case $n = 2^k$.

Example for $n = 8$

$$
\begin{tikzcd}[row sep=small]
F \bmod x^8 - 1 \arrow[r] \arrow[ddddr] & F \bmod x^4 - 1 \arrow[r] \arrow[ddr] & F \bmod x^2 - 1 \arrow[r] \arrow[dr] & F \bmod x - 1 = F(\zeta^0) \\
 & & & F \bmod x + 1 = F(\zeta^4) \\
 & & F \bmod x^2 + 1 \arrow[r] \arrow[dr] & F \bmod x - \zeta^2 = F(\zeta^2) \\
 & & & F \bmod x + \zeta^2 = F(\zeta^6) \\
 & F \bmod x^4 + 1 \arrow[r] \arrow[ddr] & F \bmod x^2 - \zeta^2 \arrow[r] \arrow[dr] & F \bmod x - \zeta = F(\zeta^1) \\
 & & & F \bmod x + \zeta = F(\zeta^5) \\
 & & F \bmod x^2 + \zeta^2 \arrow[r] \arrow[dr] & F \bmod x - \zeta^3 = F(\zeta^3) \\
 & & & F \bmod x + \zeta^3 = F(\zeta^7)
\end{tikzcd}
$$

More generally, the Cooley–Tukey algorithm reduces a transform of any length n to DFTs whose lengths are the prime factors of n.

How do we handle a DFT whose length is a **large prime**?

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Rader's algorithm (1968)

A DFT of prime length n may be reduced to a cyclic convolution of length $n - 1$, together with $O(n)$ additions in $\mathbb{C}$.

The convolution of length $n - 1$ may be evaluated by various methods (e.g. FFTs).

Example: DFT of length 5.

Given $a_0, \ldots, a_4 \in \mathbb{C}$, want to compute

$$
\begin{aligned}
&a_0 + a_1 \quad + a_2 \quad + a_3 \quad + a_4 \\
&a_0 + a_1\zeta^1 + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 \\
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where $\zeta = e^{2\pi i/5}$.

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Apart from a few additions, this is equivalent to computing

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This in turn is equivalent to computing the length-4 cyclic convolution of

$$(a_1, a_2, a_4, a_3) \quad \text{and} \quad (\zeta^3, \zeta^4, \zeta^2, \zeta^1).$$

Reductions between integer and polynomial multiplication

Can reduce integer to polynomial multiplication using **Kronecker segmentation**.

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Step 1. Rewrite u and v in base 10^3, encode them as polynomials

$$F(x) = 314x^2 + 159x + 265, \qquad G(x) = 271x^2 + 828x + 182,$$

i.e., so that $F(10^3) = u$ and $G(10^3) = v$.

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$$H(x) = F(x)G(x) = 85094x^4 + 303081x^3 + 260615x^2 + 248358x + 48230.$$

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Step 3. Substitute $x = 10^3$ to get $uv = H(10^3) = 85397341863406230$.

Reducing integer multiplication to polynomial multiplication

Application: the **first Schönhage–Strassen algorithm** (1971).

To multiply n-bit integers:

1. Rewrite integers in base 2^b where $b \approx \log n$. Encode as polynomials $F, G \in \mathbb{Z}[x]$, coefficient size b bits, degree around $n/\log n$.
2. Multiply polynomials in $\mathbb{C}[x]$ using complex FFTs, with working precision $O(\log n)$ bits. Round result to get correct product in $\mathbb{Z}[x]$.
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Complexity analysis:

$$\mathsf{M}(n) = O\Big(\tfrac{n}{\log n} \log\big(\tfrac{n}{\log n}\big)\, \mathsf{M}(\log n)\Big) = O(n\,\mathsf{M}(\log n))$$

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Step 3. Read off the desired polynomial product

$$F(x)G(x) = 85094x^4 + 303081x^3 + 260615x^2 + 248358x + 48230.$$

Application: **multiplication in $\mathbb{F}_p[x]$.**

1. Lift polynomials to $\mathbb{Z}[x]$.
2. Multiply in $\mathbb{Z}[x]$ using Kronecker substitution (i.e. via *integer* multiplication).
3. Reduce result modulo p to obtain product in $\mathbb{F}_p[x]$.

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This method is efficient provided p is not too small compared to n.

Unfortunately for fixed p this method is **inefficient** due to “zero-padding”:

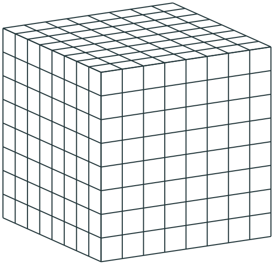
$$\mathsf{M}_p(n) = \mathsf{M}(O(n \log n)) = O(n \log^2 n).$$

Multidimensional DFTs

Example: let $F \in \mathbb{C}[x, y, z]/(x^8 - 1, y^8 - 1, z^8 - 1)$.

We may represent F with $8^3 = 512$ coefficients:

$$F = \sum_{j=0}^{7} \sum_{k=0}^{7} \sum_{l=0}^{7} F_{j,k,l} x^j y^k z^l.$$



Suppose we want to evaluate

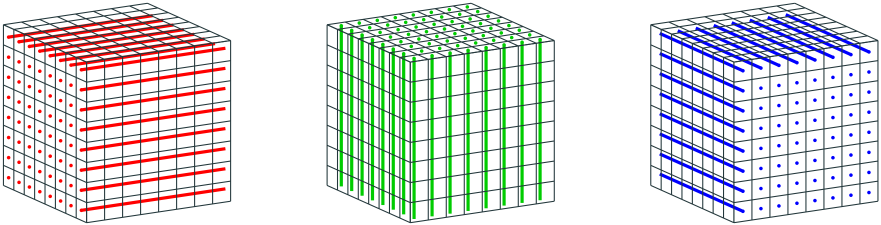
$$F(\zeta^j, \zeta^k, \zeta^l) \qquad \text{for } j, k, l = 0, \ldots, 7$$

where $\zeta = e^{2\pi i/8}$.

This is a 3-dimensional DFT of size $8 \times 8 \times 8$.

Multidimensional DFTs

Standard method for d-dimensional DFT: evaluate in each variable separately.



For a transform of size $n_1 \times \cdots \times n_d$, total cost (operations in $\mathbb{C}$) is

$$\frac{n}{n_1} O(n_1 \log n_1) + \cdots + \frac{n}{n_d} O(n_d \log n_d) = O(n \log n), \qquad n := n_1 \cdots n_d.$$

Nussbaumer's algorithm (late 1970s) instead does the following.

Suppose the input is a polynomial

$$F \in \mathbb{C}[x, y, z]/(x^8 - 1, y^8 - 1, z^8 - 1).$$

Nussbaumer's method for multidimensional DFTs

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Suppose the input is a polynomial

$$F \in \mathbb{C}[x, y, z]/(x^8 - 1, y^8 - 1, z^8 - 1).$$

First reduce modulo $z^4 - 1$ and $z^4 + 1$ just like the first step of the usual FFT:

$$F \bmod z^4 - 1 \in \mathbb{C}[x, y, z]/(x^8 - 1, y^8 - 1, z^4 - 1),$$
$$F \bmod z^4 + 1 \in \mathbb{C}[x, y, z]/(x^8 - 1, y^8 - 1, z^4 + 1).$$

We may handle the first problem recursively (in general: split in half along the "longest" dimension).

Let's concentrate on the second problem.

So now our problem is to evaluate a polynomial

$$G \in \mathbb{C}[x, y, z]/(x^8 - 1, y^8 - 1, z^4 + 1)$$

at the roots of $x^8 - 1$, $y^8 - 1$, $z^4 + 1$.

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Key observation

The roots of $z^4 + 1$ are exactly the primitive 8-th roots of unity,
so z itself behaves like a primitive 8-th root of unity in $\mathbb{C}[z]/(z^4 + 1)$.

We may evaluate x and y **at the powers of z** (instead of at the powers of ζ).

In other words, we evaluate

$$G(z^j, z^k, z) \in \mathbb{C}[z]/(z^4 + 1), \qquad j, k = 0, \ldots, 7.$$

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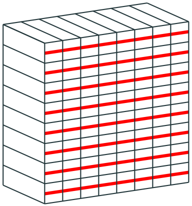
Finally, for each j and k, use the usual complex FFT to evaluate $G(z^j, z^k, z)$ at the genuine complex roots of $z^4 + 1$ (powers of ζ).

Nussbaumer's method for multidimensional DFTs



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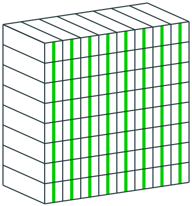

evaluate y at z^k: easy
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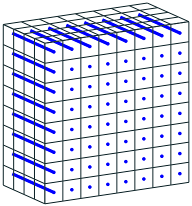

evaluate z at ζ^l: harder
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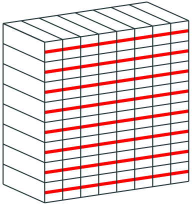

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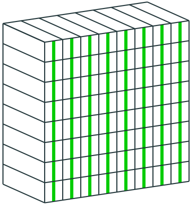

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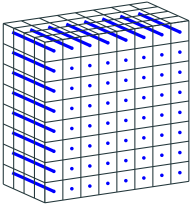

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Analysis for d-dimensional DFT **(assuming all n_i powers of two)**:

- $O(n \log n)$ additions in $\mathbb{C}$ (just like standard algorithm), but
- only $O\left(\dfrac{n \log n}{d}\right)$ multiplications in $\mathbb{C}$ **(save a factor of d)**.

Conditional $O(n \log n)$ multiplication for integers and polynomials

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$$\lceil n/46 \rceil = 2\,173\,913\,043\,479.$$

It suffices to multiply the polynomials in the ring $\mathbb{Z}[t]/(t^N - 1)$ where

$$N = 5\,509\,236\,183\,041 = p_1 p_2 p_3, \qquad p_1 = 15361,\ p_2 = 18433,\ p_3 = 19457.$$

This suffices to recover the product in $\mathbb{Z}[t]$ because $N > 2 \times 2\,173\,913\,043\,479$.

Step 2. Using CRT, there is an isomorphism (Agarwal–Cooley 1977):

$$\mathbb{Z}[t]/(t^{5\,509\,236\,183\,041} - 1) \cong \mathbb{Z}[x, y, z]/(x^{15361} - 1, y^{18433} - 1, z^{19457} - 1),$$
$$t \longmapsto xyz.$$

Can be computed efficiently in either direction (just rearrange coefficients).

So we have reduced to a 3-dimensional cyclic convolution of size $p_1 \times p_2 \times p_3$.

Step 3. To multiply in

$$\mathbb{Z}[x,y,z]/(x^{15361}-1, y^{18433}-1, z^{19457}-1),$$

we use the same strategy as the Schönhage–Strassen algorithm:

1. Compute (multidimensional) DFTs of both polynomials over $\mathbb{C}$.
2. Multiply pointwise in $\mathbb{C}$.
3. Perform inverse DFT to get approximate product in

$$\mathbb{C}[x,y,z]/(x^{15361}-1, y^{18433}-1, z^{19457}-1).$$

4. Round resulting coefficients to nearest integer.
   (Working precision throughout is a small multiple of 46 bits.)

How do we efficiently perform a DFT of size $15361 \times 18433 \times 19457$?

Nussbaumer's trick doesn't work directly, because the p_i are not powers of two.

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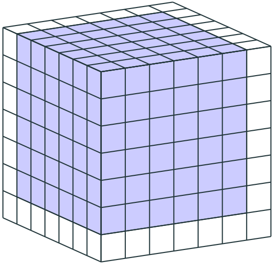
Step 4. Using multidimensional variant of Rader's trick, reduce to multiplication in

$$\mathbb{C}[x, y, z]/(x^{15360} - 1, y^{18432} - 1, z^{19456} - 1).$$



Key observation

Convolution lengths are reduced from p_i to $p_i - 1$.

I picked the primes very carefully: notice that

$$
\begin{aligned}
p_1 - 1 &= 15360 = 15 \times 2^{10}, \\
p_2 - 1 &= 18432 = 18 \times 2^{10}, \\
p_3 - 1 &= 19456 = 19 \times 2^{10}.
\end{aligned}
$$

I picked the primes very carefully: notice that

$$p_1 - 1 = 15360 = 15 \times 2^{10},$$
$$p_2 - 1 = 18432 = 18 \times 2^{10},$$
$$p_3 - 1 = 19456 = 19 \times 2^{10}.$$

Step 5. Reduce to:

- "nice" DFTs of size $2^{10} \times 2^{10} \times 2^{10}$ (use Nussbaumer), and
- "annoying" DFTs of size $15 \times 18 \times 19$.

What happens for general n? We get

- "nice" DFTs of size
$$2^k \times \cdots \times 2^k.$$
Using Nussbaumer, the first $d-1$ of the dimensions cost $O(n \log n)$. May take $d \approx 10^6$ (independently of n) to control the cost of the last dimension.

What happens for general n? We get

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- "annoying" DFTs of size

  $$\frac{p_1 - 1}{2^k} \times \cdots \times \frac{p_d - 1}{2^k}.$$

  This is where the complexity analysis becomes conditional.

To make the "annoying" DFTs cheap enough, we need to prove existence of small primes in the arithmetic progression $p = 1 \pmod{2^k}$.

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Linnik's theorem (1944)

There exists a constant $L > 1$ such that for any relatively prime integers a and $m \gg 0$, there exists a prime $p = a \pmod{m}$ with $p < m^L$.

A **Linnik constant** is a value of L for which the above statement holds.

Best published Linnik constant is currently $L = 5.18$ (Xylouris, 2011).

Linnik's theorem is **embarrassingly weak**!

Example: consider $p = 1 \pmod{2^{10}}$. The first few primes are

$$p = 12289,\ 13313,\ 15361,\ 18433,\ 19457,\ 25601,\ 37889,\ 39937, \ldots.$$

But Linnik's theorem (with the best known L) only guarantees that

$$p < (2^{10})^{5.18} \approx 4 \times 10^{15}.$$

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Under GRH, can prove that any $L > 2$ is a Linnik constant (Heath-Brown 1992).

This is still hopeless: we get

$$p < (2^{10})^2 \approx 10^6.$$

Widely-believed conjecture

Any $L > 1$ is a Linnik constant.

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Theorem (H.–van der Hoeven 2019)

If there exists a Linnik constant $L < 1 + \frac{1}{303}$, then the cost of the “annoying” DFTs can be controlled, and the algorithm sketched in this talk achieves

$$M(n) = O(n \log n).$$

We can probably weaken the bound for L a bit, but we have no idea how to get anywhere near $L = 2$.

A similar idea works for multiplying in $\mathbb{F}_q[x]$, with various additional technicalities (especially in characteristic 2):

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Conditional $O(n \log n)$ multiplication in $\mathbb{F}_q[x]$

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6. Reduce to multiplication in $\mathbb{F}_{q^s}[x_1, \ldots, x_d]/(x_1^{p_1-1} - 1, \ldots, x_d^{p_d-1} - 1)$ (Rader)

Conditional $O(n \log n)$ multiplication in $\mathbb{F}_q[x]$

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6. Reduce to multiplication in $\mathbb{F}_{q^s}[x_1, \ldots, x_d]/(x_1^{p_1-1} - 1, \ldots, x_d^{p_d-1} - 1)$ (Rader)
7. Use Nussbaumer to do synthetic FFTs in $d - 1$ dimensions, etc etc.

Conditional $O(n \log n)$ multiplication in $\mathbb{F}_q[x]$

Theorem (H.–van der Hoeven 2019)

If there exists a Linnik constant $L < 1 + 2^{-1162}$, then

$$M_q(n) = O(n \log n).$$

Can probably improve 2^{-1162}, but we don't know by how much.

Unconditional $O(n \log n)$ integer multiplication

Again take $n = 10^{14}$ bits.

As before, reduce to multiplying polynomials of degree 2 173 913 043 479 with 46-bit coefficients.

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As before, reduce to multiplying polynomials of degree 2 173 913 043 479 with 46-bit coefficients.

This time we choose primes

$$p_1 = 16381, \qquad p_2 = 16369, \qquad p_3 = 16363.$$

Notice they are all **just below $2^{14} = 16384$**.

(Easy to find such primes. No arithmetic progressions involved.)

It suffices to multiply in $\mathbb{Z}[t]/(t^N - 1)$ where

$$N = p_1 p_2 p_3 = 4\,387\,584\,457\,807 > 2 \times 2\,173\,913\,043\,479.$$

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As before, reduce to complex DFTs of size $16381 \times 16369 \times 16363$.

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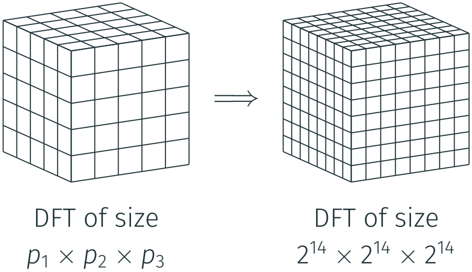
As before, reduce to complex DFTs of size $16381 \times 16369 \times 16363$.

But instead of using Rader's algorithm, we use a new technique called **Gaussian resampling** to directly reduce to a DFT of size $2^{14} \times 2^{14} \times 2^{14}$.

Then we win by using Nussbaumer's method to evaluate this last DFT.



Example: given input $u \in \mathbb{C}^{13}$, suppose we want to compute DFT $\hat{u} \in \mathbb{C}^{13}$.

Suppose however that we only know how to compute DFTs of length 16.

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Suppose however that we only know how to compute DFTs of length 16.

We will convert length 13 to length 16 via a certain **resampling map**

$$S\colon \mathbb{C}^{13} \to \mathbb{C}^{16}.$$

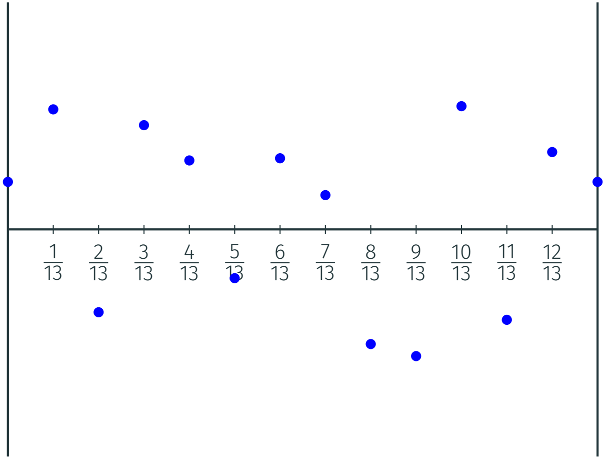
I will show how to construct S over the next few slides.

The diagram shows a typical input vector $u \in \mathbb{C}^{13}$.

For simplicity we assume $u_i \in \mathbb{R}$.

The blue points are $(\frac{i}{13}, u_i)$ for $i = 0, \ldots, 12$.

Notice the x-axis wraps around from left to right (i.e., the x-values live in $\mathbb{R}/\mathbb{Z}$).



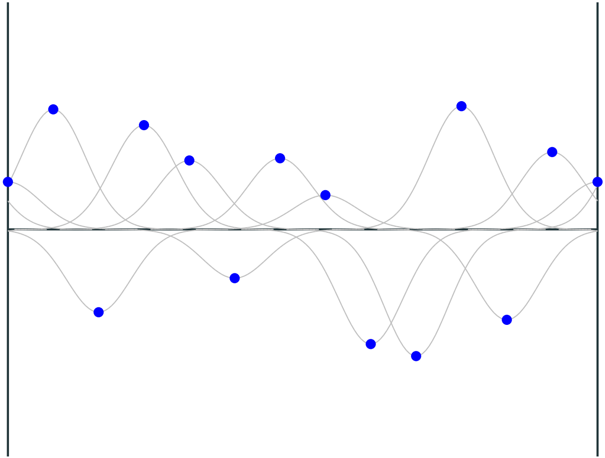
The resampling map

Draw a Gaussian curve centred around each data point.

The equation for the i-th point is

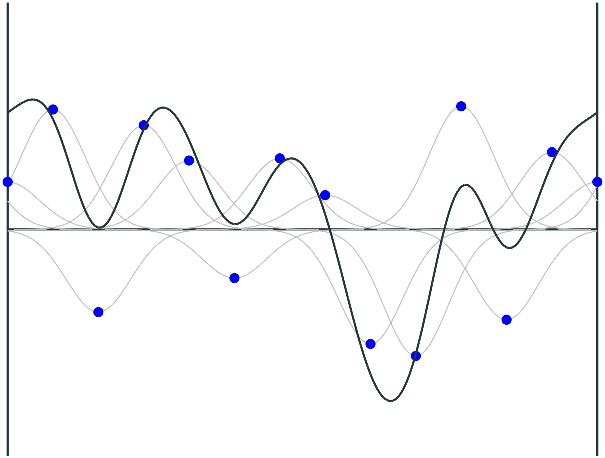
$$y = u_i e^{-13^2 (x - \frac{i}{13})^2}.$$

The "height" of the curve is u_i and the "width" is $\frac{1}{13}$.



Add up all the Gaussians to get a nice smooth 1-periodic curve:

$$f(x) = \sum_{i=0}^{12} u_i e^{-13^2(x-\frac{i}{13})^2}.$$



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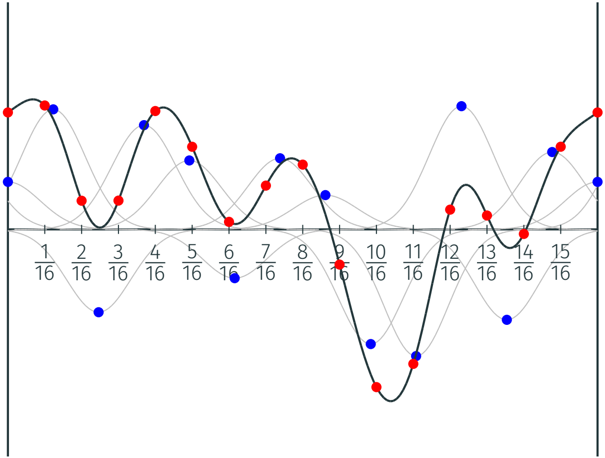
The resampled vector $v = S(u)$ is defined by evaluating $f(x)$ at 16 equally-spaced points:

$$v_j = f(\tfrac{j}{16}), \qquad j = 0, \ldots, 15.$$



$$
\begin{bmatrix}
1.000 & 0.368 & 0.018 & & & & & & & & & 0.018 & 0.368 \\
0.517 & 0.965 & 0.244 & 0.008 & & & & & & & & & 0.037 \\
0.071 & 0.677 & 0.869 & 0.151 & 0.004 & & & & & & & & 0.001 \\
0.003 & 0.127 & 0.826 & 0.729 & 0.087 & 0.001 & & & & & & & \\
 & 0.006 & 0.210 & 0.939 & 0.570 & 0.047 & 0.001 & & & & & & \\
 & & 0.014 & 0.323 & 0.996 & 0.415 & 0.023 & & & & & & \\
 & & & 0.030 & 0.465 & 0.984 & 0.282 & 0.011 & & & & & \\
 & & & 0.001 & 0.058 & 0.623 & 0.907 & 0.179 & 0.005 & & & & \\
 & & & & 0.002 & 0.105 & 0.779 & 0.779 & 0.105 & 0.002 & & & \\
 & & & & & 0.005 & 0.179 & 0.907 & 0.623 & 0.058 & 0.001 & & \\
 & & & & & & 0.011 & 0.282 & 0.984 & 0.465 & 0.030 & & \\
 & & & & & & & 0.023 & 0.415 & 0.996 & 0.323 & 0.014 & \\
 & & & & & & & 0.001 & 0.047 & 0.570 & 0.939 & 0.210 & 0.006 \\
0.003 & & & & & & & & 0.001 & 0.087 & 0.729 & 0.826 & 0.127 \\
0.071 & 0.001 & & & & & & & & 0.004 & 0.151 & 0.869 & 0.677 \\
0.517 & 0.037 & & & & & & & & & 0.008 & 0.244 & 0.965
\end{bmatrix}
$$

Matrix of resampling map $S\colon \mathbb{C}^{13} \to \mathbb{C}^{16}$.

Each “output” coordinate depends mainly on the nearby “input” coordinates.

Fun fact #1. The Fourier transform of a Gaussian is again a Gaussian. This leads to a commutative diagram

$$\begin{tikzcd}
u \arrow[r, "\text{DFT of length } s"] \arrow[d, "\substack{\text{Resample}\\ S\colon \mathbb{C}^s \to \mathbb{C}^t}"'] & \hat{u} \arrow[d, "\substack{\text{Resample}\\ \hat{S}\colon \mathbb{C}^s \to \mathbb{C}^t}"] \\
v \arrow[r, "\text{DFT of length } t"] & \hat{v}
\end{tikzcd}$$

In our example, $s = 13$ and $t = 16$.

The map $\hat{S}$ is defined almost exactly the same way as S; it differs by some straightforward scaling factors and data reindexing.

Fun fact #2. Due to the rapid decay of the Gaussians, the resampling map can be evaluated efficiently.

If the target transform length is t, the cost is

$$O(t\sqrt{\log t})$$

operations in $\mathbb{C}$ (assuming working precision $O(\log t)$ bits).

This is **asymptotically negligble** compared to $O(t \log t)$ cost of the FFT.

Fun fact #3. The resampling map is injective.

This follows more or less from the "diagonal" structure of the matrix of S.

Moreover, there is a **deconvolution algorithm** that recovers u from $v = S(u)$ using

$$O(t\sqrt{\log t})$$

operations in $\mathbb{C}$.

(Note: we do not actually prove this in the paper. For technical reasons we do something a bit different.)

Conclusion: we can compute the map $u \mapsto \hat{u}$ (a DFT of length 13) by traversing the diagram as follows:

$$
\begin{tikzcd}
u \arrow[r, dashed, "\text{DFT of length 13}"] \arrow[d, "\begin{array}{c}\text{Resample}\\ \mathbb{C}^{13} \to \mathbb{C}^{16}\end{array}"'] & \hat{u} \\
v \arrow[r, "\text{DFT of length 16}"] & \hat{v} \arrow[u, "\begin{array}{c}\text{Deconvolve}\\ \mathbb{C}^{16} \to \mathbb{C}^{13}\end{array}"']
\end{tikzcd}
$$

The cost of the vertical arrows is asymptotically negligible.

Combining a multidimensional version of Gaussian resampling with everything else from before, we get:

Theorem (H.–van der Hoeven 2019)

$$M(n) = O(n \log n).$$

Final words

Combining a multidimensional version of Gaussian resampling with everything else from before, we get:

Theorem (H.–van der Hoeven 2019)

$$M(n) = O(n \log n).$$

Unsolved problem

Can we get $M_q(n) = O(n \log n)$ unconditionally?

Unfortunately, Gaussian resampling does not seem to work over $\mathbb{F}_q$.

Is there some other way of “changing the transform length” over $\mathbb{F}_q$???

Thank you!